UK Junior Mathematical Olympiad 2012 Solutions

A1 266 $1 + 4 + 27 + 256 - (1 + 8 + 9 + 4) = 288 - 22 = 266$.

A2 200ml The remaining 40% has volume 80ml. So 10% of the volume is 20 ml and 100% of the volume is 200ml.

A3 27° Let $\angle ACD$ be $x°$. Then $\angle CDB$ is $3x°$.

Then, from the straight line ADB, $\angle ADC$ is $(180 - 3x)°$.

Consider the triangle ADC with angle sum 180°, $84 + x + (180 - 3x) = 180$, so $x = 42$.

Hence $\angle BCD = \angle DBC = \frac{1}{2}(180 - 3x)°$ which is 27°.

Alternative: Label $\angle ACD$ as $x°$, which gives $\angle CDB$ as $3x°$. Then using the property that an exterior angle is the sum of the two opposite interior angles, we have the equation $3x = x + 84$, giving $x = 42$.

Hence, as above, $\angle BCD = \angle DBC = \frac{1}{2}(180 - 3x)°$ which is 27°.

A4 8 Let the number of books bought be b and the number of magazines bought be m. Then (working in pence): $2300 = 340b + 160m$. This simplifies to $115 = 17b + 8m$ (*). But $17 \times 7 = 119$ which is greater than 115 so we know $b < 7$ and we can also see from (*) that b must be an odd number. Hence $b = 1, 3$ or 5.

If $b = 1$ then $115 = 17 + 8m$ so $m = 98/8$ which is not a whole number.

If $b = 3$ then $115 = 51 + 8m$ so $m = 64/8 = 8$.

If $b = 5$ then $115 = 85 + 8m$ so $m = 30/8$ which is not a whole number.

Therefore the only possible value of b is 3 which gives $m = 8$.

A5 1112233 An integer is divisible by 3 when the sum of its digits is divisible by 3.

Since $2 \times (1 + 2 + 3) = 12$ is divisible by 3, the digits 1, 1, 2, 2, 3, 3 are not sufficient. For the smallest possible integer, we choose an extra '1'.

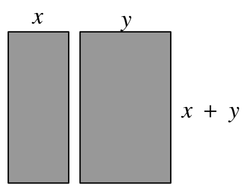
The small digits must be at the front to have the smallest integer overall. So the smallest integer made from these digits is 1112233.

A6 5cm Let the short sides of the rectangles be x cm and y cm.

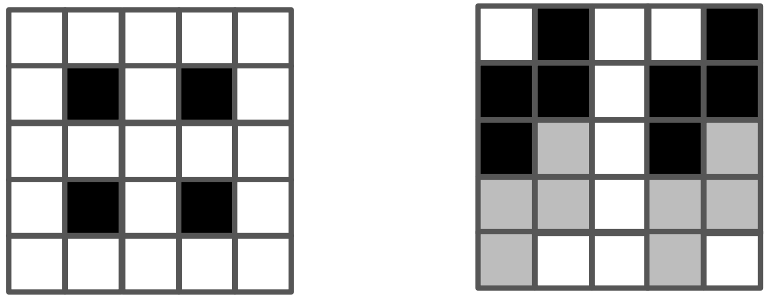
So the side of the square is $(x + y)$ cm.

Then the total perimeter is $6(x + y)$ cm.

The side of the square is therefore $30/6 = 5$ cm.



A7 4



Colour four cells of the 5 by 5 grid black, as shown in the first diagram.

Then any shape placed on the grid must cover a black cell, no matter how the shape is placed. But there are only four black cells, so the maximum number of shapes that may be placed is four. There are many ways in which this maximum can be achieved, such as the one shown on the right.

A8 128 Nine-sixteenths of the total club members are adults and seven-sixteenths are junior. So two-sixteenths of the total, the difference between the number of adults and juniors, is sixteen. Thus one-sixteenth of the total membership is 8.

The total membership is therefore $16 \times 8 = 128$.

A9 92 We require $\frac{x}{9} > \frac{71}{7}$, that is $7x > 639$, that is $x > 91\frac{2}{7}$.

We require $\frac{x}{9} < \frac{113}{11}$, that is $11x < 1017$, that is $x < 92\frac{5}{11}$.

Since x is an integer, $x = 92$.

A10 269 Suppose N is 'abc'.

Then a is at least 2, otherwise the product is at most $1 \times 9 \times 9 = 81$, and so does not have three digits.

Consider numbers of the form '2bc'. Then b is at least 6, otherwise the product is at most $2 \times 5 \times 9 = 90$, and so does not have three digits.

Consider numbers of the form '26c'. Then c is at least 9, otherwise the product is at most $2 \times 6 \times 8 = 96$, and so does not have three digits.

Now the product of the digits of 269 is $2 \times 6 \times 9 = 108$, so 269 is the smallest value of N.

B1 There was an old woman who lived in a shoe. She had 9 children at regular intervals of 15 months. The oldest is now six times as old as the youngest. How old is the youngest child?

Solution

Let the youngest child be aged x years. Then the oldest child's age is $x + 8 \times \frac{15}{12} = x + 10$. So $x + 10 = 6x$. Then $5x = 10$. Thus $x = 2$, so the youngest child is 2.

B2 Anastasia thinks of a positive integer, which Barry then doubles. Next, Charlie trebles Barry's number. Finally, Damion multiplies Charlie's number by six. Eve notices that the sum of these four numbers is a perfect square. What is the smallest number that Anastasia could have thought of?

Solution

Let Anastasia's integer be a. Then the four numbers are a, $2a$, $6a$ and $36a$. The sum of these numbers is $45a$. Now $45 = 3 \times 3 \times 5 = 9 \times 5$. So $45a$ being a square means that $5a$ must be a square because 9 is already a square. So to be the smallest, $5a$ must be 25. Thus the smallest a is 5.

B3 Mr Gallop has two stables which each initially housed three ponies. His prize pony, Rein Beau, is worth £250 000. Usually Rein Beau spends his day in the small stable, but when he wandered across into the large stable, Mr Gallop was surprised to find that the average value of the ponies in each stable rose by £10 000. What is the total value of all six ponies?

Solution

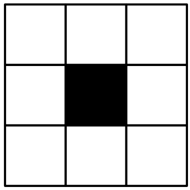
At the start, let the value of the three ponies in the small stable be £$(s + 250\,000)$ and the value of the other three ponies in the large stable be £l.

Then $\frac{s}{2} - \frac{s + 250\,000}{3} = 10\,000$ and so $s = 560\,000$. Also $\frac{l + 250\,000}{4} - \frac{l}{3} = 10\,000$ and so $l = 630\,000$.

Therefore the total value of all six ponies is £$(s + l + 250\,000) = $ £1 440 000.

B4 An irregular pentagon has five different interior angles each of which measures an integer number of degrees. One angle is 76°.

The other four angles are three-digit integers which fit one digit per cell across and down into the grid on the right.



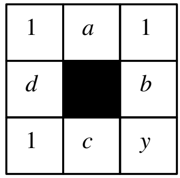
In how many different ways can the grid be completed?

Solution

The sum of the interior angles of a pentagon is 540°.

The remaining four angles therefore add to 464°.

The integer 1 must appear in both top corners and in the left-hand bottom corner. We see that we need to find integers a, b, c, d and y all less than or equal to 9 so that '1a1' + '1by' + '1cy' + '1d1' = 464. So $1 + y + y + 1 = 4$ or 14 since y is at most 9.



Consider $2 + 2y = 4$, then $y = 1$ and we now need to find different single digit integers a, b, c and d so that $a + b + c + d = 6$. The only possible values of a, b, c, d are 0, 1, 2, 3. There are four ways of choosing a, then three ways of choosing b, then two ways of choosing c and only one way then left for d. In this case, there are $4 \times 3 \times 2 \times 1 = 24$ ways of completing the grid.

Consider $2 + 2y = 14$, then $y = 6$ and we now need to find $a + b + c + d = 5$ with $a \neq d$ and $b \neq c$. The only possible sets of integers are 0, 0, 1, 4; 0, 0, 2, 3; 0, 1, 1, 3 and 0, 1, 2, 2. Each of these sets can be used in 8 ways. For example,

a	0	0	0	0	1	1	4	4
b	0	0	1	4	0	4	0	1
c	1	4	0	0	4	0	1	0
d	4	1	4	1	0	0	0	0

making a total of another 32 ways.

Altogether, there are 56 possible ways of completing the grid.

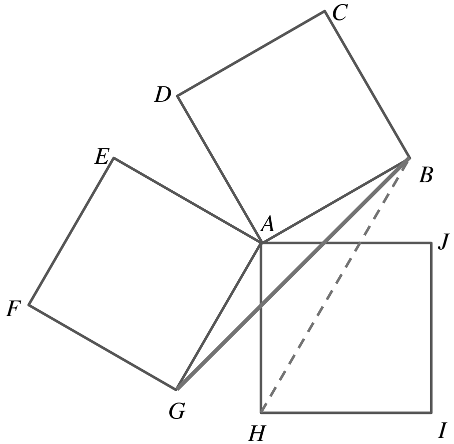
B5 Three identical, non-overlapping squares $ABCD$, $AEFG$, $AHIJ$ (all labelled anticlockwise) are joined at the point A, and are 'equally spread' (so that $\angle JAB = \angle DAE = \angle GAH$). Calculate $\angle GBH$.

Solution

Since the squares are 'equally spread' $\angle JAB = \angle DAE = \angle GAH = 30°$. Hence $\angle GAB = \angle GAH + \angle HAJ + \angle JAB = 30° + 90° + 30° = 150°$. Triangle GAB is isosceles since $GA = GB$.

Therefore $\angle GBA = \angle BGA = 15°$.

Similarly $\angle HAB = \angle HAJ + \angle JAB = 90° + 30° = 120°$. Triangle BAH is isosceles since $BA = HA$. Therefore $\angle AHB = \angle ABH = 30°$ which gives $\angle GBH = \angle ABH - \angle GBA = 30° - 15° = 15°$.



Observe that the diagram is not the only valid arrangement so the answer of 15° is not unique.

B6 The integer 23173 is such that

(a) every pair of neighbouring digits, taken in order, forms a prime number;

and (b) all of these prime numbers are different.

What is the largest integer which meets these conditions?

Solution

No two-digit prime ends in 5 or an even digit, so that 2, 4, 6, 8 or 5 can only appear as the first digit of the required number.

There are ten two-digit prime numbers with two odd digits that do not include 5. We may list them in a table, putting those with the same first digit in the same row, and those with the same second digit in the same column:

11	13	17	19
31		37	
71	73		79
		97	

Now any 12-digit number which contains all of these ten primes and has a digit 2, 4, 6, 8, or 5 at the front will clearly be larger than any failing to do so. And no number of the required form can have more than 12 digits. Let us assume that we can find a number of this form, that is, containing all ten primes in the table and with first digit 2, 4, 6, 8, or 5, and try to construct the largest such number.

Notice that any digit x not at an end of the number corresponds to two primes ax and xb, so that the digit x is both the first digit of a prime and the second digit of a prime.

From the table, we see that the digit 1 occurs four times as a first digit, and only three times as a second digit, so that if we are to use all these primes, one of those of the form '1_' does not appear as part of a pair from the table. The only possible way this can happen is for the number to start 'd1_', where d is 5 or even.

Similarly, for all the primes containing a digit 9 to appear, the number has to end '_9'.

Now the largest prime of the form 'd1', where d is 5 or even, is 61. And the largest number of the form '61e' with '1e' prime is 619. This leaves 79 as the only possible last pair of digits. So we now have a number of the form 619 _ _ _ _ _ _ _ 79. From the table the only option now is to use 97 at the start, giving a number of the form 619 7 _ _ _ _ _ _79.

To have the largest possible answer, there is only one choice for the digit before the 7 at the end, so we have
619 7 _ _ _ _ _ 179.

We may continue to add the largest possible digit to the end of those at the start. The sequence continues:

619	73 _	_ _ _	179
619	737	_ _ _	179
619	737	1 _ _	179
619	737	13 _	179
619	737	131	179

The final number uses all ten primes from the table, together with the largest possible first digit of the numbers that do so. When constructing the number, we have also used the largest possible next digit at each stage.
Therefore 619 737 131 179 is the largest number of the required form.