STEP Support Programme

Assignment 2

Warm-up

1 **(i)** Simplify $(2x-3)^2-(x-1)^2$, giving your answer in factorised form. Check your answer by evaluating it for $x=1$ and $x=2$.

(ii) Simplify

$$\frac{x}{x^2-y^2}-\frac{y}{(x-y)^2}-\frac{1}{x+y}\,.$$

Hence find the possible values of x and y for which $\dfrac{x}{x^2-y^2}-\dfrac{y}{(x-y)^2}-\dfrac{1}{x+y}=0$.

(iii) Show that

$$\sqrt{1+x^2}\;-x=\frac{1}{\sqrt{1+x^2}\;+x}\,.$$

Deduce that if x is very large, then $\sqrt{1+x^2}-x$ is approximately equal to $\dfrac{1}{2x}$.

(iv) Simplify $(x^2-\sqrt{2}\,x+1)(x^2+\sqrt{2}\,x+1)$. Hence find the solutions to $x^4+1=0$.

Your answer will involve $\sqrt{-1}$ (written as i).

Preparation

2 **(i)** Sketch the line $y = x + 1$ for $-2 \leqslant x \leqslant 2$.

What is the greatest value of $x + 1$ in this range?

(ii) Sketch the line $y = -2x + c$ for $-2 \leqslant x \leqslant 2$.

Show that the greatest value of $-2x + c$ in this range is $4 + c$. What is the least value?

(iii) Sketch $y = mx + 1$ for $-2 \leqslant x \leqslant 2$ in the cases $m > 0$, $m = 0$ and $m < 0$.

What are the greatest and least values of $mx + 1$ in each case?

(iv) Sketch the curve (parabola) $y = (x - 1)^2$ for $-2 \leqslant x \leqslant 2$.

What are the greatest and least values of $(x - 1)^2$ in this range?

Be careful here: the minimum value is **not** at one of the end points.

(v) Sketch the curve $y = (x - 3)^2$. What are the greatest and least values of $(x - 3)^2$ for $-2 \leqslant x \leqslant 2$?

(vi) Write the expression $x^2 - 8x + 21$ in the form $(x + a)^2 + b$. Hence sketch the curve $y = x^2 - 8x + 21$ and find the greatest and least values of $x^2 - 8x + 21$ in the range $0 \leqslant x \leqslant 5$.

(vii) Sketch the curve $y = x^2 + 2kx$ for $-2 \leqslant x \leqslant 2$, where $-2 < k < 2$.

What are the greatest and least values of $x^2 + 2kx$ for $-2 \leqslant x \leqslant 2$?

What would your answers be if $k > 2$?

Use the same techniques as in part **(vi)** to help you sketch the curve.

The STEP question

3 **(i)** Find the greatest and least values of $bx + a$ for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases $b > 0$, $b = 0$ and $b < 0$.

(ii) Find the greatest and least values of $cx^2 + bx + a$, where $c \geqslant 0$, for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases that can arise for different values of b and c.

Discussion

This question has some features that are very typical of STEP questions.

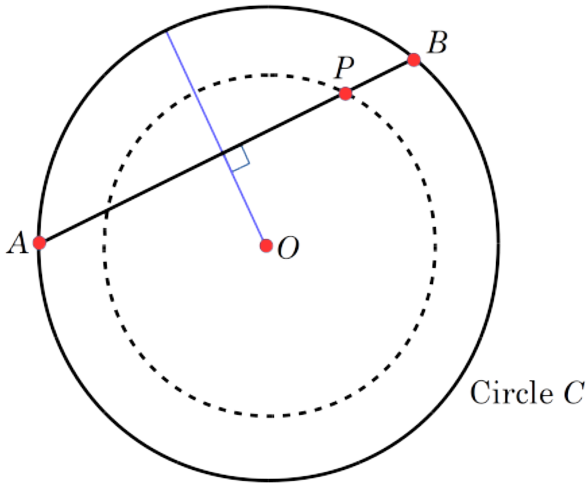
First is the use of letters, a, b and c in this case, rather than numbers to make the equations less specific. The correct term for these letters is *parameters*, whereas x is a *variable*. In each case you will have to give your answers in terms of the given parameters.

Second, although you are not told to draw sketches, you should do so: it is much easier to work on the different cases if you have sketches in front of you. Think back to question **2** to help you.

Warm down

4 The diagram shows a circle C with centre O, and a rod AB the ends of which can slide round the circle C (so that AB is a chord of C). The radius of the circle is R and the length of the rod is $2a$.

As the rod slides round C the point P, which is a fixed distance b from the centre of the rod, traces out a circle with centre O of radius r.



Show that the area between the two circles is $\pi(a^2 - b^2)$.

The surprising thing about this result is that it is independent of the radius of C (assuming that it is greater than a) and depends only on the length of the stick and the position of P on the stick. It doesn't matter how big the circle is, the area between the two circles is always $\pi(a^2 - b^2)$.

If you had known that the answer was independent of R, can you think of an easy way (by choosing R) of obtaining the result?

Even more surprising is the fact that the result holds even when C is not a circle, but is any closed curve round which the rod can slide smoothly. This is Holditch's theorem, proved in about 1840, and not much seen until it was used as a STEP question in 2010.

The lengths R and r are given here for your convenience: they are required for the calculation but do not appear in the answer. In a STEP question, it would have been up to you to decide what is needed for the calculation. And you might have had to draw the diagram yourself.