STEP Support Programme

Assignment 1

Warm-up

1 **(i)** Simplify $\sqrt{50} + \sqrt{18}$.

(ii) Express $(3 + 2\sqrt{5})^3$ in the form $a + b\sqrt{5}$ where a and b are integers.

(iii) Expand and simplify

$$(1 - \sqrt{2} + \sqrt{6})^2 \,.$$

(iv) **(a)** Expand and simplify $(1 + \sqrt{2})^2$.

(b) Find all real values of x that satisfy

$$x^2 + \frac{4}{x^2} = 12 \,.$$

Leave your answers in the form of surds.

Preparation

2 **(i)** Solve the equation:

$$\frac{2}{x+3} + \frac{1}{x+1} = 1 \,.$$

(ii) Find the value(s) of b for which the following equation has a single (repeated) root.

$$9x^2 + bx + 4 = 0 \,.$$

(iii) Find the range of (real) values of c for which the following equation has no real roots:

$$3x^2 + 5cx + c = 0 \,.$$

Probably the safest way of dealing with inequalities is to sketch a graph.

The STEP question

3 In this question a and b are distinct, non-zero real numbers, and c is a real number.

(i) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

has two distinct real solutions.

(ii) Show that, if $c \neq 1$, the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

has exactly one real solution if

$$c^2 = -\frac{4ab}{(a-b)^2} \, .$$

Show that this condition can be written

$$c^2 = 1 - \left(\frac{a+b}{a-b}\right)^2$$

and deduce that it can only hold if $0 < c^2 \leqslant 1$.

Discussion

This question requires fluent algebraic manipulation and a clear head. Your layout needs to be clear — the mantra "One equal sign per line, all equal signs aligned." is a good one to follow. Be careful when using the formula "$b^2 - 4ac$" for the discriminant when one or more of the letters a b or c are defined differently in the question (as in this question). It is safer to write "$B^2 - 4AC = 0$" or "$b'^2 - 4a'c' = 0$" so that you don't get confused.

Where the answer is given ("Show that") you must be particularly careful to show every step of the argument in sufficient detail to convince yourself (and any examiners reading it) that your argument is correct and complete. Some people consider it mathematically uncouth to start with what you want to show and work backwards to the given starting point — however you can always work backwards on a scrap piece of paper and then write up your final solution reversing the steps (as long as they are truly reversible!). Or you could work forwards and backwards and meet in the middle, provided that you then construct a coherent proof.

Note that the last part uses the work "deduce" — this means that you must use the previous part and also implies that not much further working is required; however your explanations must still be clear and logical. Remember that the information given about a, b and c in the very first line of the question holds throughout the question.

Warm down

4 A Minister and a Bishop were having a cup of tea. There was a knock at the door, and three bell ringers entered the room. After introductions, the Bishop asked the Minister how old the bell ringers were.

"Well," the Minister said, knowing the Bishop had a penchant for numerical puzzles, "if you multiplied their three ages together, you'd get 2,450. But if you added them, you'd get twice your age."

"Hmm," the Bishop muttered, after several moments' thought. "I haven't enough information to solve that."

"It may help, my dear Bishop," offered the Minister, "to know that I am older than anyone else here in the room."

"Yes, indeed it would," replied the Bishop. "Now I know their ages."

The question is: How old is the Minister?

You may assume that all ages are integers.

This rather subtle puzzle is intended mainly for amusement, but also to initiate a quick discussion about the peculiarly English pursuit of Change Ringing. The bells are numbered from lightest (and highest pitched) to heaviest such as 1, 2, 3, 4, 5, 6. The lightest bell is called the Treble and the heaviest the Tenor. When they ring in descending order it is known as "rounds".

A "method" on six bells is a sequence of the form (......)(......)(......)(......) where the dots in each bracket represent one bell being rung and within each bracket none of the bells is repeated. The ordering or permutation within each bracket is different; no permutation is repeated.

For example, he first two brackets (changes) of "Plain Bob Minor" are (2 1 4 3 6 5) and (2 4 1 6 3 5).

There is a further rule: any bell can only change position by at most one place from one bracket to the next (each stroke). This is a rather practical rule. Bells can be a ton or more in weight and once they get going it is hard to change their rhythm significantly.

If all the possible permutations are rung (once and once only!) then this is known as an "extent" or "full peel". The length of an extent on 5 bells is $5! = 120$ and takes about 5 minutes to ring. An extent on 8 bells has only ever been rung once which was in 1963 in Loughborough: it took just under 18 hours. An extent on the 12 bells of St Mary's in Cambridge would take over 30 years to ring.