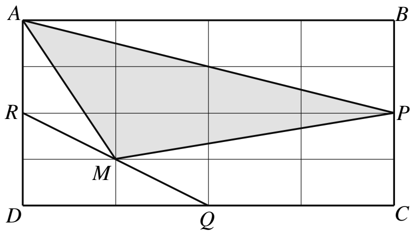
16. 891 Since $m = 10^{99} - 1$, we have $m^2 = \left(10^{99} - 1\right)^2 = 10^{198} - 2 \times 10^{99} + 1$ $= 999\ldots 9998000\ldots 001$ where there are 98 nines and 98 zeroes. Therefore the sum of the digits is $98 \times 9 + 8 + 1 = 891$.

17. 21



Dividing rectangle *ABCD* into 16 equal parts, as shown in the diagram above, demonstrates that the area of $\triangle APM = 12 - \frac{1}{2} \times 3 - \frac{1}{2} \times 3 - \frac{1}{2} \times 8 = 5$ parts. Therefore the area of $\triangle APM$ is $\frac{5}{16}$ of the area of rectangle *ABCD* so $m + n = 21$.

18. 4 There are six different numbers that can be formed with digits a, b and c. The sum of these six numbers is

$$
\begin{aligned}
&(100a + 10b + c) + (100a + 10c + b) + (100b + 10a + c) \\
&+ (100b + 10c + a) + (100c + 10a + b) + (100c + 10b + a) \\
&= 200(a + b + c) + 20(a + b + c) + 2(a + b + c) \\
&= 222(a + b + c) = 1554
\end{aligned}
$$

so $a + b + c = 7$. Thus the only possibility for a, b and c is 1, 2 and 4 so $c = 4$.

19. 3 From $\left(a + \frac{1}{a}\right)^2 = 6$ we have $a + \frac{1}{a} = \sqrt{6}$ since $a > 0$. Therefore $\left(a + \frac{1}{a}\right)^3 = (\sqrt{6})^3$, which gives $a^3 + 3a^2 \times \frac{1}{a} + 3a \times \frac{1}{a^2} + \frac{1}{a^3} = 6\sqrt{6}$ and so $N\sqrt{6} + 3\left(a + \frac{1}{a}\right) = 6\sqrt{6}$. This means that $N = 3$.

20. 17 From $f(x^2 + 1) \equiv x^4 + 4x^2 \equiv (x^2 + 1)^2 + 2(x^2 + 1) - 3$ we deduce that $f(w) \equiv w^2 + 2w - 3$ and hence that $f(x^2 - 1) \equiv (x^2 - 1)^2 + 2(x^2 - 1) - 3 \equiv x^4 - 2x^2 + 1 + 2x^2 - 2 - 3 \equiv x^4 - 4$. This means $a = 1$, $b = 0$ and $c = -4$. Therefore the value of $a^2 + b^2 + c^2$ is 17.

An alternative solution is to realise that $f(x^2 + 1) \equiv [(x^2 + 1) + 1]^2 - 4$. So $f(x^2 - 1) \equiv [(x^2 - 1) + 1]^2 - 4 \equiv x^4 - 4$. This gives the same value for $a^2 + b^2 + c^2$.

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 2nd December 2011

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

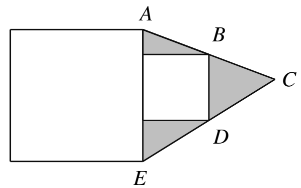
1. 71 Since the entries are different, no entry can be 1, and the smallest total will come from using the integers 2, 3 and 4 on the top line of the diagram. Treating an ordering and its reverse as the same, since they give the same total, we can arrange these in the orders 2, 3, 4 or 2, 4, 3 or 3, 2, 4. Of these, 3, 2, 4 gives the smallest overall total of 71.

2. 390 Over the years 2007 to 2010, the school accepted $325 \times 4 = 1300$ students. The mean for 2007 to 2011 is 4% higher than 325, which is 338. This means that $338 \times 5 = 1690$ students were accepted over the years 2007 to 2011. Therefore 390 students were accepted in 2011.

3. 16 On the first pass, all 200 people receive a pound coin. On the second pass, only people in even numbered positions receive a coin. On the nth pass, people receive a coin if n divides the number representing their position. Now 120 is divisible by 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60 and 120 so the 120th person receives 16 pound coins.

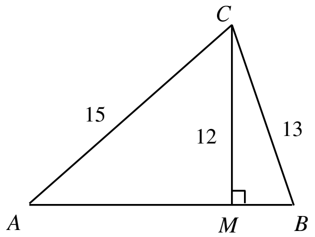
[*Note that, in general, if the prime decomposition of an integer*, X, *is* $p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots p_n^{a_n}$ *then the number of divisors of* X *is* $(a_1 + 1)(a_2 + 1)(a_3 + 1)\ldots(a_n + 1)$.]

4. 100 Using the labelling shown, we see that $\triangle ACE$ and $\triangle BCD$ are similar and have lengths in the ratio 2:1. Because the height of $\triangle ACE$ is 10 + the height of $\triangle BCD$, the height of $\triangle ACE$ is 20 and its area is $\frac{1}{2} \times 20 \times 20 = 200$. The area of the smaller square is 100 so the shaded area is $200 - 100 = 100$.



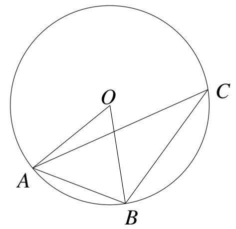
5. 36 The square and cube of an integer end in the same digit if, and only if, the integer itself ends in 0, 1, 5 or 6. The two-digit numbers with this property can have any tens digit from 1 to 9 so there are $4 \times 9 = 36$ such two-digit numbers.

6. **84** The perpendicular height from any side of an acute-angled triangle is always less than the length of either of the other two sides so the height is 12. Thus we have the situation shown in the diagram alongside.



Using Pythagoras' theorem in ΔACM, we obtain $AM = 9$ and in ΔCMB we get $MB = 5$. This means that $AB = 14$ and the area of triangle ABC is $\frac{1}{2} \times 14 \times 12 = 84$.

7. **30** Let the centre of the circle be O. Join A to C and O to A and B, as shown in the diagram alongside.



In ΔAOB, $AO = OB$ since they are both radii, and we are given that AB has length equal to the radius so $AB = AO = OB$ and ΔAOB is equilateral. Hence $\angle AOB = 60°$. Since the angle at the centre is twice the angle on the circumference, $\angle ACB$ is 30°.

8. **13** Let the original price, in pence, be p.
The new price is 4% more than the original so, working in pence,

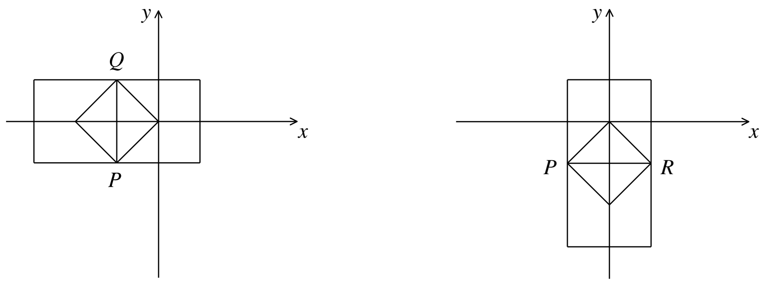
$$100n = \frac{104}{100} \times p,$$

which may be rearranged to

$$n = \frac{13}{1250} \times p.$$

Now n is an integer and 1250 is not divisible by 13, so p is divisible by 1250. The smallest value of n will be when $p = 1250$, which means n is 13.

9. **5**



Let P be $(-1, -1)$ and suppose that the x-axis is a line of symmetry. Then $Q\,(-1, 1)$ is a vertex of the square since it is the reflection of the vertex P in the x-axis. Hence PQ is either an edge or a diagonal of the square. In the first case there are two possible squares and in the second case there is one, as shown in the first figure.

Similarly, when the y-axis is a line of symmetry there are three possible squares. However, one of these is the same as before, so in all there are exactly five squares possible.

10. **4** By expanding the brackets, we obtain

$$\left(\sqrt{8+2\sqrt{7}} - \sqrt{8-2\sqrt{7}}\right)^2 = 8 + 2\sqrt{7} - 2\sqrt{\left(8+2\sqrt{7}\right)\left(8-2\sqrt{7}\right)} + 8 - 2\sqrt{7}$$

$$= 16 - 2\sqrt{64-28} = 16 - 2\sqrt{36} = 16 - 12 = 4.$$

11. **60** Area ΔABC = area ΔACD + area ΔBCD + area ΔABD

$$= \tfrac{1}{2} \times e \times 5 + \tfrac{1}{2} \times f \times 12 + \tfrac{1}{2} \times g \times 13 = \tfrac{1}{2}\left(5e + 12f + 13g\right).$$

But ΔABC has sides 5, 12 and 13, hence it is a right-angled triangle and so has area $\frac{1}{2} \times 5 \times 12 = 30$. Therefore $5e + 12f + 13g = 60$.

12. **40** From the information given about those who voted, we can conclude:

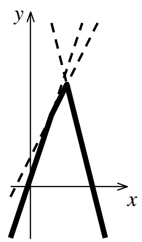
		Eaten broccoli?	
		Yes	No
Voted Broccoli Party?	Yes	y	0
	No	x	$9x$

where x, $9x$ and y are the appropriate percentages of those who voted.
We are given that $x + y = 46$ and, since the table includes everyone, we also have $x + y + 9x = 100$. So $9x = 54$ and $x = 6$. Therefore $y = 40$ and so the percentage that voted for the Broccoli Party is 40%.

13. **65** Suppose n represents the number of increments of €5 above (or below, if n is negative) the selling price of €75. Then the number of sweaters sold is $100 - 20n$ and the profit made, in Euros, is $\left((75 + 5n) - 30\right)(100 - 20n) = (45 + 5n)(100 - 20n) = 100(5 - n)(9 + n)$. So the profit is $100\left(49 - (n + 2)^2\right)$ and is a maximum when $n = -2$. This gives a sale price of €65.

14. **75** Using Pythagoras' theorem firstly in ΔABC and then in ΔACE we get $AC = 20$ and $AE = 25$. It follows that ΔABC is similar to ΔACE as the corresponding sides are in the same ratio. Therefore, $\angle BAC = \angle CAE$. Also $\angle BAC = \angle ACF$, using alternate angles, so $\angle CAF = \angle ACF$ and ΔAFC is isosceles. Let M be the mid-point of AC and join M to F. This gives two more right-angled triangles, ΔAMF and ΔCMF, also similar to ΔABC. Thus $\dfrac{MF}{MA} = \dfrac{BC}{BA}$ which gives $MF = \dfrac{15}{2}$. Therefore the area of ΔACF is $\dfrac{1}{2} \times \dfrac{15}{2} \times 20 = 75$.

15. **10**



The diagram shows the dashed lines with equations $y = 3x + 1$, $y = 2x + 3$ and $y = -4x + 24$. The solid lines form the graph of the function given in the question. We can see that the maximum value of $f(x)$ occurs when $y = 2x + 3$ crosses $y = -4x + 24$.

At this point $y = 10$, therefore the maximum value of $f(x)$ is 10.