# OXFORD UNIVERSITY

## MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

## WEDNESDAY 4 NOVEMBER 2009

## Time allowed: $2\frac{1}{2}$ hours

*For candidates applying for Mathematics, Mathematics & Statistics,*
*Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy*

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in **BLOCK CAPITALS**.

<u>**NOTE**</u>: Separate sets of instructions for both candidates and test supervisors are provided, which should be read carefully before beginning the test.

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**NAME:**

**TEST CENTRE:**

**OXFORD COLLEGE (if known):**

**DEGREE COURSE:**

**DATE OF BIRTH:**

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**Signature of Invigilator** _____

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**1.** For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

|   | (a) | (b) | (c) | (d) |
|---|-----|-----|-----|-----|
| **A** |  |  |  |  |
| **B** |  |  |  |  |
| **C** |  |  |  |  |
| **D** |  |  |  |  |
| **E** |  |  |  |  |
| **F** |  |  |  |  |
| **G** |  |  |  |  |
| **H** |  |  |  |  |
| **I** |  |  |  |  |
| **J** |  |  |  |  |

**A.** The smallest value of

$$I\left(a\right) = \int_0^1 \left(x^2 - a\right)^2 \, \mathrm{d}x,$$

as $a$ varies, is

(a) $\dfrac{3}{20}$,     (b) $\dfrac{4}{45}$,     (c) $\dfrac{7}{13}$,     (d) 1.

**B.** The point on the circle

$$x^2 + y^2 + 6x + 8y = 75,$$

which is closest to the origin, is at what distance from the origin?

(a) 3,     (b) 4,     (c) 5,     (d) 10.

**Turn Over**

**C.** Given a real constant $c$, the equation

$$x^4 = (x - c)^2$$

has four real solutions (including possible repeated roots) for

(a) $\quad c \leqslant \dfrac{1}{4}$, (b) $\quad -\dfrac{1}{4} \leqslant c \leqslant \dfrac{1}{4}$, (c) $\quad c \leqslant -\dfrac{1}{4}$, (d) $\quad$ all values of $c$.

**D.** The smallest positive integer $n$ such that

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n \geqslant 100,$$

is

(a) $\quad$ 99, (b) $\quad$ 101, (c) $\quad$ 199, (d) $\quad$ 300.

**E.** In the range $0 \leqslant x < 2\pi$, the equation

$$2^{\sin^2 x} + 2^{\cos^2 x} = 2$$

(a) has 0 solutions;
(b) has 1 solution;
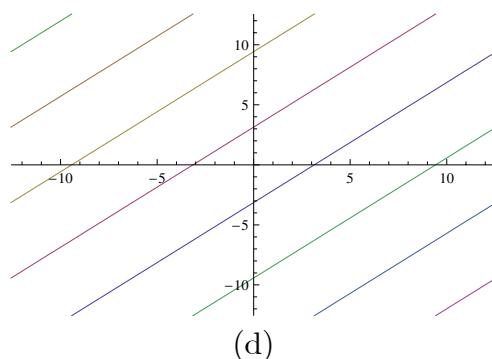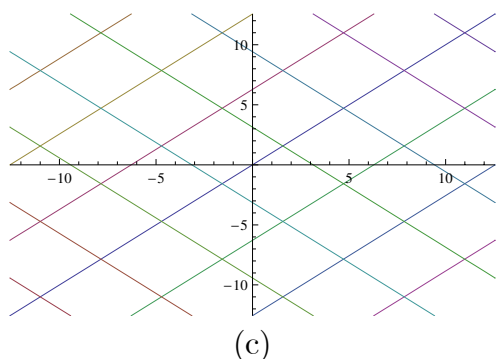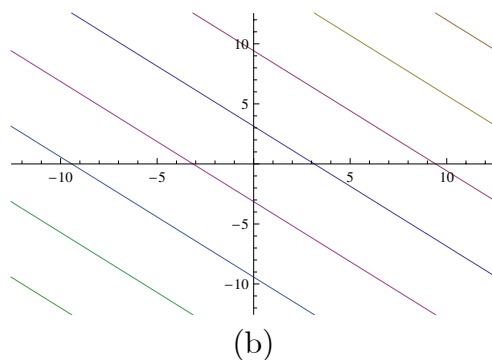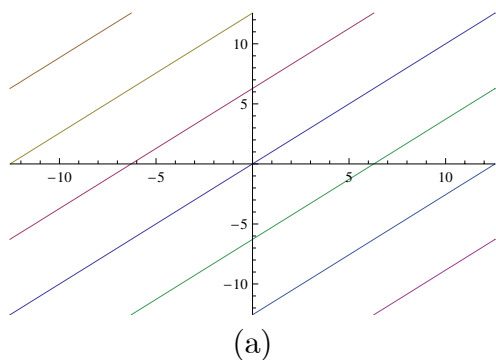(c) has 2 solutions;
(d) holds for all values of $x$.

**F.** The equation in $x$

$$3x^4 - 16x^3 + 18x^2 + k = 0$$

has four real solutions
(a) when $-27 < k < 5$;
(b) when $5 < k < 27$;
(c) when $-27 < k < -5$;
(d) when $-5 < k < 0$.

**G.** The graph of all those points $(x, y)$ in the $xy$-plane which satisfy the equation $\sin y = \sin x$ is drawn in



(a)



(b)



(c)



(d)

**H.** When the trapezium rule is used to estimate the integral

$$\int_0^1 2^x \, \mathrm{d}x$$

by dividing the interval $0 \leqslant x \leqslant 1$ into $N$ subintervals the answer achieved is

(a) $\quad \dfrac{1}{2N}\left\{1 + \dfrac{1}{2^{1/N} + 1}\right\},$

(b) $\quad \dfrac{1}{2N}\left\{1 + \dfrac{2}{2^{1/N} - 1}\right\},$

(c) $\quad \dfrac{1}{N}\left\{1 - \dfrac{1}{(2^{1/N} - 1)}\right\},$

(d) $\quad \dfrac{1}{2N}\left\{\dfrac{5}{2^{1/N} + 1} - 1\right\}.$

**I.** The polynomial
$$n^2 x^{2n+3} - 25nx^{n+1} + 150x^7$$

has $x^2 - 1$ as a factor
(a) for no values of $n$;
(b) for $n = 10$ only;
(c) for $n = 15$ only;
(d) for $n = 10$ and $n = 15$ only.

**J.** The number of *pairs* of *positive integers* $x, y$ which solve the equation
$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is

(a) 0,     (b) $2^6$,     (c) $2^9 - 1$,     (d) $2^{10} + 2$.

**Turn Over**

**2.** For **ALL APPLICANTS.**

A list of real numbers $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 1$, $x_2 = 3$ and then for $n \geqslant 3$ by

$$x_n = 2x_{n-1} - x_{n-2} + 1.$$

So, for example,

$$x_3 = 2x_2 - x_1 + 1 = 2 \times 3 - 1 + 1 = 6.$$

(i) Find the values of $x_4$ and $x_5$.

(ii) Find values of real constants $A, B, C$ such that $\underline{\text{for } n = 1, 2, 3,}$

$$x_n = A + Bn + Cn^2. \qquad (*)$$

(iii) Assuming that equation $(*)$ holds true $\underline{\text{for all } n \geqslant 1}$, find the smallest $n$ such that $x_n \geqslant 800$.

(iv) A second list of real numbers $y_1, y_2, y_3, \ldots$ is defined by $y_1 = 1$ and

$$y_n = y_{n-1} + 2n$$

Find, explaining your reasoning, a formula for $y_n$ which holds for $n \geqslant 2$.

What is the approximate value of $x_n/y_n$ for large values of $n$?

**3.**

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

*Computer Science* **applicants should turn to page 14.**

For a positive whole number $n$, the function $f_n(x)$ is defined by

$$f_n(x) = \left( x^{2n-1} - 1 \right)^2.$$

(i) On the axes provided opposite, sketch the graph of $y = f_2(x)$ labelling where the graph meets the axes.

(ii) On the same axes sketch the graph of $y = f_n(x)$ where $n$ is a large positive integer.

(iii) Determine

$$\int_0^1 f_n(x) \, dx.$$
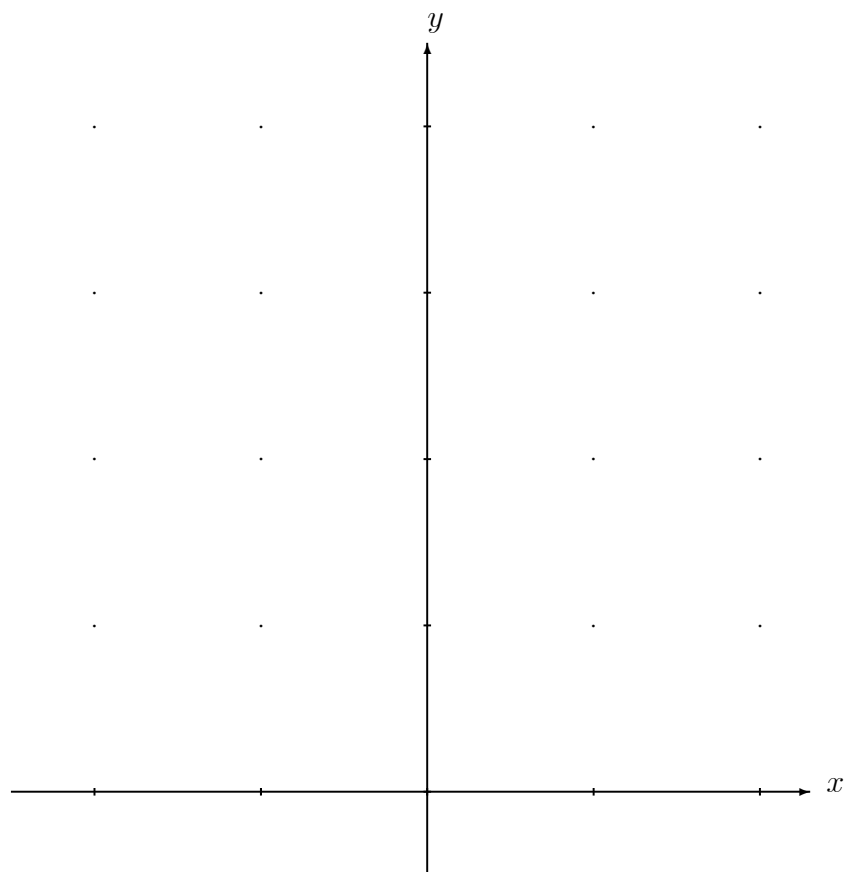
(iv) The *positive* constants $A$ and $B$ are such that

$$\int_0^1 f_n(x) \, dx \leqslant 1 - \frac{A}{n+B} \quad \text{for all } n \geqslant 1.$$

Show that

$$(3n - 1)(n + B) \geqslant A(4n - 1)n,$$

and explain why $A \leqslant 3/4$.

(v) When $A = 3/4$, what is the smallest possible value of $B$?

**4.**

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

*Mathematics & Computer Science* **and** *Computer Science* **applicants should turn to page 14.**

As shown in the diagram below: $C$ is the parabola with equation $y = x^2$; $P$ is the point $(0, 1)$; $Q$ is the point $(a, a^2)$ on $C$; $L$ is the normal to $C$ which passes through $Q$.
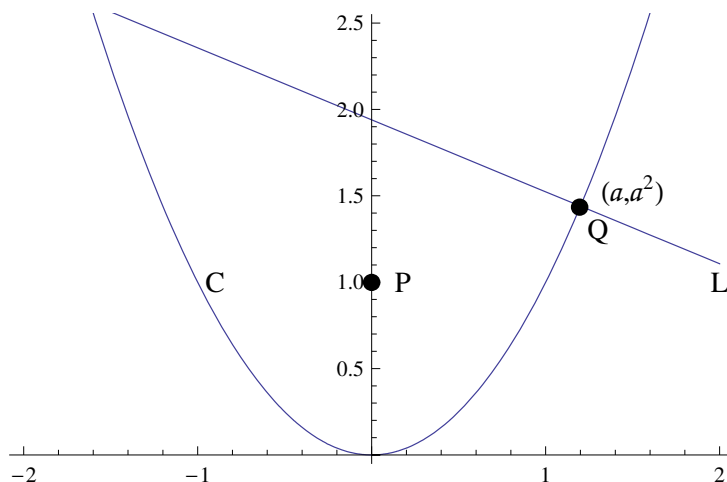


(i) Find the equation of $L$.

(ii) For what values of $a$ does $L$ pass through $P$?

(iii) Determine $|QP|^2$ as a function of $a$, where $|QP|$ denotes the distance from $P$ to $Q$.

(iv) Find the values of $a$ for which $|QP|$ is smallest.

(v) Find a point $R$, in the $xy$-plane but not on $C$, such that $|RQ|$ is smallest for a unique value of $a$. Briefly justify your answer.

**5.** For **ALL APPLICANTS.**

Given an $n \times n$ grid of squares, where $n > 1$, a *tour* is a path drawn within the grid such that:

- along its way the path moves, horizontally or vertically, from the centre of one square to the centre of an adjacent square;

- the path starts and finishes in the same square;

- the path visits the centre of every other square just once.

For example, below is a tour drawn in a $6 \times 6$ grid of squares which starts and finishes in the top-left square.
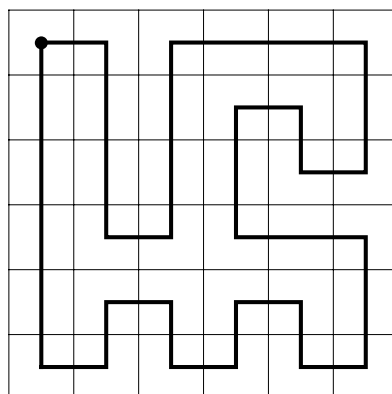


For parts (i)-(iv) it is assumed that $n$ is *even*.

(i) With the aid of a diagram, show how a tour, which starts and finishes in the top-left square, can be drawn in any $n \times n$ grid.

(ii) Is a tour still possible if the start/finish point is changed to the centre of a different square? Justify your answer.

Suppose now that a robot is programmed to move along a tour of an $n \times n$ grid. The robot understands two commands:

- command $R$ which turns the robot clockwise through a right angle;

- command $F$ which moves the robot forward to the centre of the next square.

The robot has a program, a list of commands, which it performs in the given order to complete a tour; say that, in total, command $R$ appears $r$ times in the program and command $F$ appears $f$ times.

(iii) Initially the robot is in the top-left square pointing to the right. Assuming the first command is an $F$, what is the value of $f$? Explain also why $r + 1$ is a multiple of 4.

(iv) Must the results of part (iii) still hold if the robot starts and finishes at the centre of a different square? Explain your reasoning.

(v) Show that a tour of an $n \times n$ grid is not possible when $n$ is odd.

**6.**

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

(i) Alice, Bob, and Charlie make the following statements:

| | |
|---|---|
| Alice: | Bob is lying. |
| Bob: | Charlie is lying. |
| Charlie: | $1 + 1 = 2$. |

Who is telling the truth? Who is lying?. Explain your answer.

(ii) Now Alice, Bob, and Charlie make the following statements:

| | |
|---|---|
| Alice: | Bob is telling the truth. |
| Bob: | Alice is telling the truth. |
| Charlie: | Alice is lying. |

What are the possible numbers of people telling the truth? Explain your answer.

(iii) They now make the following statements:

| | |
|---|---|
| Alice: | Bob and Charlie are both lying. |
| Bob: | Alice is telling the truth or Charlie is lying (or both). |
| Charlie: | Alice and Bob are both telling the truth. |

Who is telling the truth and who is lying on this occasion? Explain your answer.

**7.** For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

Consider sequences of the letters M, X and W. *Valid* sequences are made up according to the rule that an M and a W can never be adjacent in the sequence. So M, XMXW, and XMMXW are examples of valid sequences, whereas the sequences MW and XWMX are not valid.

(i) Clearly, there are 3 valid sequences of length 1. List all valid sequences of length 2.

(ii) Let $g(n)$ denote the number of valid sequences of length $n$. Further, let $m(n)$, $x(n)$, $w(n)$ denote the number of valid sequences of length $n$ that start with an M, an X, a W respectively.

Explain why

$$
\begin{aligned}
m(n) &= w(n), \\
m(n) &= m(n-1) + x(n-1) \quad \text{for } n > 1, \\
x(n) &= 2m(n-1) + x(n-1) \quad \text{for } n > 1,
\end{aligned}
$$

and write down a formula for $g(n)$ in terms of $m(n)$ and $x(n)$.

Hence compute $g(3)$, and verify that $g(4) = 41$.

(iii) Given a sequence using these letters then we say that it is *reflexive* if the following operation on the sequence does not change it: reverse the letters in the sequence, and then replace each occurrence of M by W and vice versa. So MXW, WXXM and XWXMX are reflexive strings, but MXM and XMXX are not. Let $r(n)$ be the number of valid, reflexive sequences of length $n$.

If a sequence is reflexive and has odd length, what must the middle letter be? Explain your answer.

Hence, show that
$$
r(n) = \begin{cases} x\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd,} \\ x\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}
$$

**End of Last Question**