15. 186 The possible quadratic equations are $(x-80)(x-1)=0$, $(x-40)(x-2)=0$, $(x-20)(x-4)=0$, $(x-16)(x-5)=0$, $(x-10)(x-8)=0$. These equations give the values of b as 81, 42, 24, 21 and 18 respectively. The sum of the possible values of b is 186.

16. 343 Expanding $(a+b)^4$ gives $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. We can rearrange this to give $a^4+b^4=(a+b)^4-(4a^3b+6a^2b^2+4ab^3)$ which can be written in the form $a^4+b^4=(a+b)^4-(4ab(a^2+b^2)+6a^2b^2)$. Now using $a^2+b^2=(a+b)^2-2ab$ we can write $a^4+b^4=(a+b)^4-(4ab((a+b)^2-2ab)+6(ab)^2)$, that is $a^4+b^4=(a+b)^4-4ab(a+b)^2+2(ab)^2$. Substituting $a+b=5$ and $ab=3$, we obtain $a^4+b^4=5^4-(4\times3(5^2-2\times3)+6\times3^2)$ which reduces to 343.

Note: For a quicker solution, observe that $a^2+b^2=(a+b)^2-2ab=5^2-2\times3=19$ so that $a^4+b^4=(a^2+b^2)^2-2a^2b^2=19^2-2(ab)^2=361-18=343$.

17. 223 Let the smallest of the ten original numbers be x. The sum of all ten numbers is $x+(x+1)+(x+2)+\ldots+(x+9)$, which equals $10x+45$.

Let the number removed be $x+a$ where $0\leqslant a\leqslant 9$. Removing this number from the ten original numbers leaves 2012 so $10x+45-(x+a)=2012$, that is, $9x=1967+a$. Dividing by 9, we get $x=218\frac{5}{9}+\frac{a}{9}$. Since x is an integer, $a=4$. Hence $x=219$ and the number removed, $x+a$, is 223.

18. 96 The numbers 1, 2, 3, 4, 5 and 6 can be paired uniquely (1 with 4, 2 with 5, and 3 with 6) so that the difference between the numbers in each pair is 3. Any of the six numbers can be placed in position F but once that has been done, the partner of this number must be placed in position C as, otherwise, the square containing F would be adjacent to a square containing a number that differs by 3. Any of the remaining four numbers may be placed at position A, but then the number placed at B must be one of the two numbers from the pair that has so far been left unused. Finally, the two unused numbers may be positioned at D and E, in either order. Thus the number of possible arrangements is $6\times4\times2\times2$, which is 96.

19. 116 Let the lengths of the sides of the rectangle be a and b, where $a<b$. For the rectangle to be cut from the square, $b<20$. But $ab=36$ and a and b are integers, so the greatest possible value of b is 18. Note that cutting the rectangle so that it shares a corner with the original square and so two sides of the rectangle form part of two sides of the original square would still leave the remaining shape with a perimeter of 80. Cutting the rectangle from the square with one side of length a taken to be part of one of the sides of the square will give a perimeter of $80+2b$. Similarly, if the side of length b is taken to be part of one of the sides of the square, the perimeter of the new shape will be $80+2a$. For the largest perimeter, it is clearly better to do the former since a is less than b. Now $80+2b$ is largest when b is largest, so putting $b=18$ we obtain the largest perimeter which is 116.

20. 341 If the smallest element was 1 then the largest element would be 10 (to give a sum of 11). Of the remaining possible elements, 2 could be included or not, 3 could be included or not and so on. This means that there are 2^8 possible subsets with smallest element 1 and largest element 10. Similarly there are 2^6 possible subsets with smallest element 2 and largest element 9, there are 2^4 possible subsets with smallest element 3 and largest element 8, there are 2^2 possible subsets with smallest element 4 and largest element 7, and there is one possible subset with smallest element 5 and largest element 6. In total this means there are $2^8+2^6+2^4+2^2+1=341$ possible subsets.

SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 30th November 2012

Organised by the United Kingdom Mathematics Trust

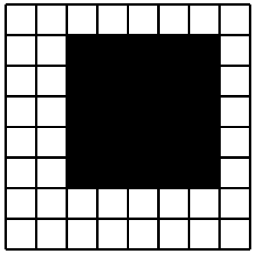
SOLUTIONS

2012 Senior Kangaroo Solutions

1. **1** In general, each zero at the end of an integer arises because, in the prime factorisation of the integer, there is a factor of 2 and a factor of 5 that can be paired to give a factor of 10. For example, $38\,000 = 2^4 \times 5^3 \times 19$ so 2 and 5 may be paired three times giving a factor of 1000. The product of the first 2012 prime numbers only contains a single factor of 2 and a single factor of 5 so there is only one zero at the end.

2. **206** Let the increase from one term to the next be i. From $225\frac{1}{2}$ to 284 the increase is $3i$ so $284 - 225\frac{1}{2} = 3i$. Therefore $3i = 58\frac{1}{2}$ and hence $i = 19\frac{1}{2}$. To find the value of a, we need to decrease $225\frac{1}{2}$ by i which gives $a = 206$.

3. **16** The diagram shows a region R, say, that certainly contains the region A, and has the same perimeter as A. We claim that R is the region with the largest possible area with these properties and so is B, the region required. To see this, observe that adding any number of extra grid squares to R will only increase the perimeter and so will not give a region of the type required. Therefore, the maximum number of additional grid squares that can be added is 16.



4. **2** For Sylvia to be certain that Peter is telling the truth, she must check that the card showing the letter E has an even number on the other side and that the card showing the number 7 does not have a vowel on the other side. The other three cards do not need to be checked since K is not a vowel and cards with even numbers on one side may have any letter on the other side.

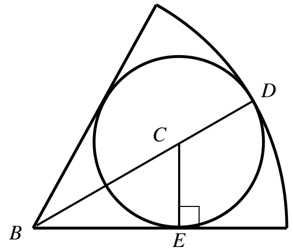
5. **9** For the weights of the two equally thick pendants to be the same, the area of the annulus must be equal to the area of the disc. The area of the annulus is $\pi \times 6^2 - \pi \times 4^2$ which is 20π. Since the area of a disc of radius r is πr^2, we have $\pi r^2 = 20\pi$ and so $r = \sqrt{20} = 2\sqrt{5}$. Thus the diameter of the second pendant is $2 \times 2\sqrt{5}$ which is $4\sqrt{5}$. So $a + b = 4 + 5 = 9$.

6. **4** We have $4^{xy} = (4^x)^y = 9^y = 256$. However $256 = 4^4$ so $xy = 4$.

7. **2** Let the single-digit number be k. Since the remainder is 5, k is larger than 5. Also k is a single-digit number so k is less than 10. Thus k is 6, 7, 8 or 9. However, the remainder when 1001 is divided by k is 5 so 996 is a multiple of k. Since 996 is not divisible by 7, 8 and 9 we conclude that $k = 6$. Finally, 2012 leaves a remainder of 2 when divided by 6.

8. **962** Adding the two given equations, we get $2a = 74$, which means $a = 37$. Substituting this value into the first equation we obtain $b + c = 15$. Therefore one of b and c is even and the other is odd. Since b and c are both prime, one of them is 2. This means the other is 13. Thus the product required is $2 \times 13 \times 37$, which is 962.

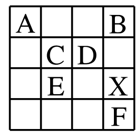
9. **5** Label the diagram as shown. Let the radius of the small circle be r, therefore $r = CD = CE$. We are told that the ratio of the radius of the sector to the radius of the small circle is 3 : 1 so $BD = 3 \times CD = 3r$. Therefore $BC = 2r$. Since triangle BCE is right-angled with two known sides, we have angle CBE is 30° and the area of the sector is a sixth of the area of a circle of radius $3r$. The ratio of the area of the sector to the area of the small circle is $\frac{1}{6}\pi(3r)^2 : \pi r^2$ which simplifies to $\frac{9}{6}\pi r^2 : \pi r^2$, this is, 3 : 2 in its simplest form. So $p + q = 5$.



10. **15** Each team played 15 matches so the maximum possible score for any one team is 15 points. Since the total score for a team must be an integer, the gap between consecutive team scores is also an integer. With 16 teams and no negative total scores, the gap is 1 point. This means the total scores are 15 points, 14 points, 13 points , …, 1 point, 0 points. Thus the team in first place scored 15 points, winning all their matches.

11. **99** Let the number of girls in the choir last year be x. This means there were $x + 30$ boys in the choir last year. This year there are 20% more girls, that is, $1.2x$ girls, and 5% more boys, that is, $1.05(x + 30)$ boys. The overall number in the choir this year is 10% more, that is, $1.1(x + x + 30)$. Putting these together, we get $1.1(2x + 30) = 1.2x + 1.05(x + 30)$. Multiplying out the brackets, we obtain $2.2x + 33 = 1.2x + 1.05x + 31.5$ and hence $15 = 0.05x$, so $x = 30$. The number in the choir this year is $2.2(2x + 30)$ which is therefore $2.2(2 \times 30 + 30)$. So there are 99 choir members this year.

12. **4** In the diagram, the squares labelled A, B, C, D, E and F all need to switch colour (from black to white or white to black, as appropriate). This could take as few as 3 exchanges, if the squares were distributed helpfully. Unfortunately, this is not the case. For example, each of A and F can only be paired with B. This means that it is impossible to use just three exchanges. So if we can perform the required switches in four exchanges, this must be the minimum number needed. This can be done in a number of ways, for example, exchange A and B, exchange C and D, exchange E and X, exchange X and F.



13. **400** Let the radius of each of the small circles be r. So the area of the whole window is $\pi(2r)^2$, which is $4\pi r^2$. Therefore $4\pi r^2 = 4(\mathrm{R} + \mathrm{G} + \mathrm{B})$ and $\pi r^2 = \mathrm{R} + \mathrm{G} + \mathrm{B}$. Now consider the area of one of the small circles. This is πr^2 but is also $\mathrm{R} + 2\mathrm{G}$. Equating these expressions for πr^2 gives $\mathrm{R} + \mathrm{G} + \mathrm{B} = \mathrm{R} + 2\mathrm{G}$, which simplifies to $\mathrm{B} = \mathrm{G}$. Thus the area of the blue glass is equal to the area of the green glass.

Alternative: Note that the area of the large circle is equal to the area of four smaller circles since $\pi(2r)^2 = 4 \times \pi r^2$. But

the area of the window = the area of the 4 smaller circles − 4G + 4B.

Hence B = G.

14. **10** Let $BX = p$, $BY = q$ and $BZ = r$. Using Pythagoras' theorem in ΔBXY, ΔBXZ and ΔBYZ respectively, we obtain the equations: $p^2 + q^2 = 81$, $p^2 + r^2 = 64$ and $q^2 + r^2 = 55$. Adding all three equations, we get $2(p^2 + q^2 + r^2) = 81 + 64 + 55$ that is $p^2 + q^2 + r^2 = 100$. By applying Pythagoras' theorem to ΔBXZ and ΔAXZ, we find that the length of the diagonal XA is $\sqrt{p^2 + q^2 + r^2}$, which is $\sqrt{100} = 10$.