UK Junior Mathematical Olympiad 2013

Organised by The United Kingdom Mathematics Trust

Tuesday 11th June 2013

RULES AND GUIDELINES : READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, measuring instruments and squared paper is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work. Write in blue or black pen or pencil.

   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 45 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you can't do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like π, fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

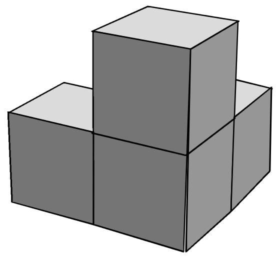
Section A

A1 What is the value of $\sqrt{3102 - 2013}$?

A2 For how many three-digit positive integers does the product of the digits equal 20?

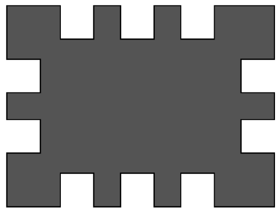
A3 The solid shown is made by gluing together four 1 cm × 1 cm × 1 cm cubes.

What is the total surface area of the solid?



A4 What percentage of $\frac{1}{4}$ is $\frac{1}{5}$?

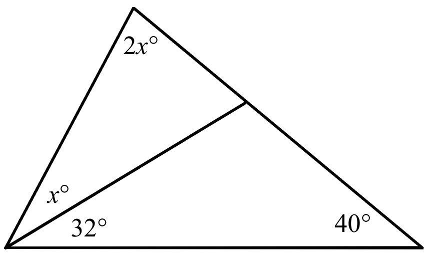
A5 Sue has a rectangular sheet of paper measuring 40 cm × 30 cm. She cuts out ten squares each measuring 5 cm × 5 cm, as shown. In each case, exactly one side of the square lies along a side of the rectangle and none of the cut-out squares overlap.



What is the perimeter of the resulting shape?

A6 I want to write a list of integers containing two square numbers, two prime numbers, and two cube numbers. What is the smallest number of integers that could be in my list?

A7 Calculate the value of x in the diagram shown.



A8 The area of a square is 0.25 m^2. What is the perimeter of the square, in metres?

A9 Each interior angle of a quadrilateral, apart from the smallest, is twice the next smaller one. What is the size of the smallest interior angle?

A10 A cube is made by gluing together a number of unit cubes face-to-face.
The number of unit cubes that are glued to exactly four other unit cubes is 96.

How many unit cubes are glued to exactly five other unit cubes?

Section B

Your solutions to Section B will have a major effect on your JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1 How many numbers less than 2013 are both:

(i) the sum of two consecutive positive integers; **and**

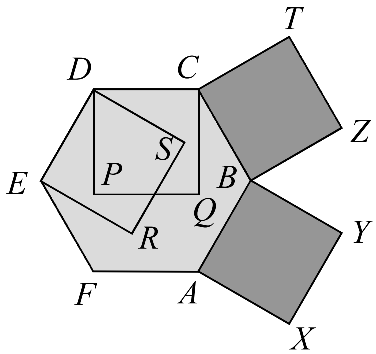
(ii) the sum of five consecutive positive integers?

B2 Pippa thinks of a number. She adds 1 to it to get a second number. She then adds 2 to the second number to get a third number, adds 3 to the third to get a fourth, and finally adds 4 to the fourth to get a fifth number.

Pippa's brother Ben also thinks of a number but he subtracts 1 to get a second. He then subtracts 2 from the second to get a third, and so on until he too has five numbers.

They discover that the sum of Pippa's five numbers is the same as the sum of Ben's five numbers. What is the difference between the two numbers of which they first thought ?

B3 Two squares *BAXY* and *CBZT* are drawn on the outside of a regular hexagon *ABCDEF*, and two squares *CDPQ* and *DERS* are drawn on the inside, as shown.



Prove that $PS = YZ$.

B4 A regular polygon P with n sides is divided into two pieces by a single straight cut. One piece is a triangle T, the other is a polygon Q with m sides.

How are m and n related?

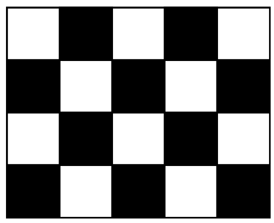
B5 Consider three-digit integers N with the two properties:

(a) N is not exactly divisible by 2, 3 or 5;

(b) no digit of N is exactly divisible by 2, 3 or 5.

How many such integers N are there?

B6 On the 4×5 grid shown, I am only allowed to move from one square to a neighbouring square by crossing an edge. So the squares I visit alternate between black and white. I have to start on a black square and visit each black square exactly once. What is the smallest number of white squares that I have to visit? Prove that your answer is indeed the smallest.



(If I visit a white square more than once, I only count it as one white square visited).

UK Junior Mathematical Olympiad 2013 Solutions

A1 33 $3102 - 2013 = 1089 = 9 \times 121 = 3^2 \times 11^2 = 33^2$. Therefore $\sqrt{3102 - 2013} = \sqrt{33^2} = 33$.

A2 9 First, we need to find triples of digits whose product is $20 = 2^2 \times 5$.
The only possible triples are $\{1, 4, 5\}$ and $\{2, 2, 5\}$.
There are 6 possible ways of ordering the digits: $\{1, 4, 5\}$.
There are 3 possible ways of ordering the digits: $\{2, 2, 5\}$.
Therefore the total number of 3-digit numbers for which the product of the digits is equal to 20 is 9.

A3 18 cm^2 The surface area of four such cubes arranged separately is $4 \times 6\text{ cm}^2 = 24\text{ cm}^2$. However, in this solid, there are three pairs of faces that overlap and so do not contribute to the surface area of the solid.
Therefore, the total surface area is $(24 - 3 \times 2 \times 1)\text{ cm}^2 = 18\text{ cm}^2$.

Alternatively, a bird's eye view from each of six directions has surface area 3 cm^2. So the total surface area is $6 \times 3 = 18\text{ cm}^2$.

A4 80% Note that $\frac{1}{5} : \frac{1}{4} = \frac{4}{20} : \frac{5}{20} = 4 : 5 = 80 : 100$. Hence $\frac{1}{5}$ is 80% of $\frac{1}{4}$.
Alternatively $\frac{1}{5} \div \frac{1}{4} = \frac{4}{5}$; and $\frac{4}{5} \times 100 = 80$. Therefore $\frac{1}{5}$ is 80% of $\frac{1}{4}$.

A5 240 cm The perimeter of the original paper is $40 + 40 + 30 + 30 = 140\text{ cm}$.
Each cut-out square adds 10 cm to the perimeter.
So the final perimeter is $140 + 10 \times 10 = 240\text{ cm}$.

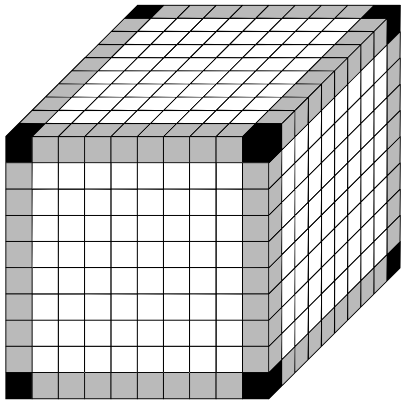
A6 4 A prime number cannot be a square or cube.
Hence there must be at least 4 numbers in the list.
We can find a list with two sixth powers (i.e. both squares and cubes) and two prime numbers e.g. $1^6, 2, 3, 2^6$ or $2^6, 3^6, 5, 7$ (where 2, 3, 5, 7 are all prime).
So the smallest number of integers in my list is 4.

A7 36 Since the angles in a triangle add up to 180°, we have $2x + (x + 32) + 40 = 180$.
This simplifies to $3x + 72 = 180$, which has the solution $x = 36$.

A8 2 m The length of each side equals 0.5 m since $0.5^2 = 0.25$.
Hence the perimeter is $4 \times 0.5 = 2\text{ m}$.

A9 **24°** Let the angles of the quadrilateral be $x°$, $2x°$, $4x°$ and $8x°$.
The sum of angles in a quadrilateral is 360°.
Thus $15x = 360$ which gives $x = 24$.

A10 **384** Let the side length of the large cube be n.



On each edge of the large cube, there are $n - 2$ cubes glued to exactly 4 other cubes, shown shaded grey.

So in total there are $12(n - 2)$ cubes glued to exactly 4 other cubes.

Therefore $12(n - 2) = 96$ which gives $n = 10$.

On each face of the large cube, there are 8^2 cubes glued to exactly 5 other cubes, which are unshaded.

So in total, there are $6 \times 64 = 384$ cubes glued to exactly five other unit cubes.

B1 How many numbers less than 2013 are both:

(i) the sum of two consecutive positive integers; **and**

(ii) the sum of five consecutive positive integers?

Solution

A number satisfies condition (i) if and only if it is of the form

$$n + (n + 1) = 2n + 1$$

for $n \geqslant 1$, i.e. it is an odd number greater than or equal to 3.

A number satisfies condition (ii) if and only if it is of the form

$$(m - 2) + (m - 1) + m + (m + 1) + (m + 2) = 5m$$

for some $m \geqslant 3$, i.e. it is a multiple of 5 greater than or equal to 15.

So a number satisfies both conditions if and only if it is of the form $5p$ with p an odd number and $p \geqslant 3$; i.e. $p = 2q + 1$ for $q \geqslant 1$.

Now $5(2q + 1) \leqslant 2013$ implies that $q \leqslant 200$. So there are 200 such numbers. satisfying both conditions.

B2 Pippa thinks of a number. She adds 1 to it to get a second number. She then adds 2 to the second number to get a third number, adds 3 to the third to get a fourth, and finally adds 4 to the fourth to get a fifth number.

Pippa's brother Ben also thinks of a number but he subtracts 1 to get a second. He then subtracts 2 from the second to get a third, and so on until he too has five numbers.

They discover that the sum of Pippa's five numbers is the same as the sum of Ben's five numbers. What is the difference between the two numbers of which they first thought ?

Solution

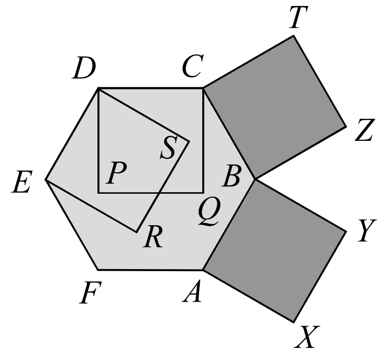
Let Pippa's original number be p and Ben's be b.

Then $p + (p + 1) + (p + 1 + 2) + (p + 1 + 2 + 3) + (p + 1 + 2 + 3 + 4)$
$= b + (b - 1) + (b - 1 - 2) + (b - 1 - 2 - 3) + (b - 1 - 2 - 3 - 4)$.

This simplifies first to $5p + 20 = 5b - 20$ and then to $8 = b - p$.

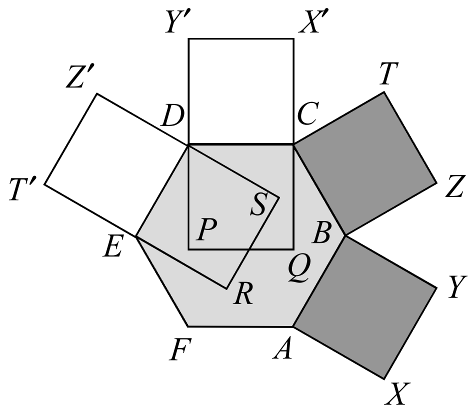
Hence the difference between the original numbers is 8.

B3 Two squares *BAXY* and *CBZT* are drawn on the outside of a regular hexagon *ABCDEF*, and two squares *CDPQ* and *DERS* are drawn on the inside, as shown.



Prove that $PS = YZ$.

Solution 1



Draw squares $EDZ'T'$ and $DCX'Y'$ on the outside of the hexagon.

Since *ABCDEF* is regular, angles *EDC* and *ABC* are both 120° and also angles EDZ', CDY', *CBZ* and *ABY* are all right angles so $\angle Y'DZ' = \angle YBZ = 60°$. Also the lengths of the sides of the four squares are all equal as they are equal to the sides of the regular hexagon. Thus triangles $DY'Z'$ and *BYZ* are congruent (SAS) and hence $Z'Y' = ZY$.

Now compare triangles $Y'DZ'$ and *PDS*.

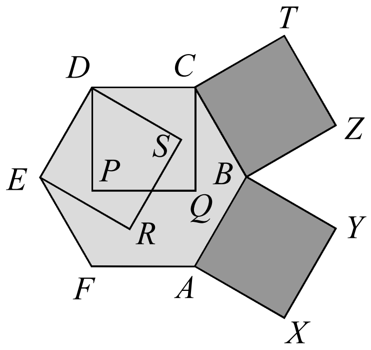
$Y'D = PD$ and $Z'D = DS$. (These are the same length as the sides of the hexagon.)

Angle $Y'DZ'$ = Angle *PDS* as they are vertically opposite angles.

So triangle $Y'DZ'$ is congruent to triangle *PDS*.

So $PS = Z'Y' = ZY$.

Solution 2



In a regular hexagon, an interior angle is 120°. In a square an interior angle is 90°.

Consider ΔBZY. $BZ = CB$ as both are sides of the square $BZTC$. $CB = AB$ as both are sides of the regular hexagon. $AB = BY$ as both are sides of the square $BAXY$. Therefore $BZ = BY$.

Since angles at a point total 360°, it follows that $90° + 120° + 90° + \angle ZBY = 360°$ and so $\angle ZBY = 60°$.

Therefore ΔBZY is an isosceles triangle with an angle of 60° between the equal sides and so is an equilateral triangle.

In a similar way, consider ΔDPS. $DP = DC$ as both are sides of the square $DPQC$; $DC = DE$ as both are sides of the regular hexagon and $DE = DS$ as both are sides of the square $DERS$. Hence $DP = DS$.

$\angle EDC = 120°$ and $\angle PDC = 90°$ hence $\angle EDP = 30°$. $\angle EDC = 120°$ and $\angle EDS = 90°$ hence $\angle SDC = 30°$. $\angle EDC = \angle EDP = \angle PDS = \angle SDC$ so $\angle PDS = 60°$.

Therefore ΔPDS is an isosceles triangle with an angle of 60° between the equal sides and so is an equilateral triangle.

Also $DP = DC = CB = BZ$ so the sides of the two equilateral triangles are the same length.

Therefore ΔDPS and ΔBYZ are exactly the same size (*called congruent triangles*).

Therefore $PS = YZ$.

B4 A regular polygon *P* with *n* sides is divided into two pieces by a single straight cut. One piece is a triangle *T*, the other is a polygon *Q* with *m* sides.

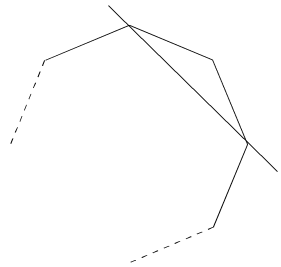
How are *m* and *n* related?

Solution

There are three possible ways in which one straight cut can create a triangle.

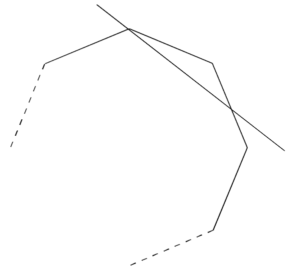
Case 1: The straight cut goes through two vertices of the polygon.

Then $m = n - 1$.



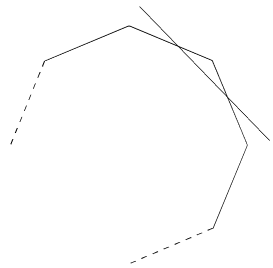
Case 2: The straight cut goes through exactly one vertex of the polygon.

Then $m = n$.



Case 3: The straight cut goes through no vertices of the polygon.

Then $m = n + 1$.



B5 Consider three-digit integers *N* with the two properties:

(a) *N* is not exactly divisible by 2, 3 or 5;

(b) no digit of *N* is exactly divisible by 2, 3 or 5.

How many such integers N are there?

Solution

Condition (b) means that each digit of *N* must be either 1 or 7 (since 0, 2, 4, 6, 8 are divisible by 2; 0, 5 are divisible by 5 and 3, 6, 9 are divisible by 3).

A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

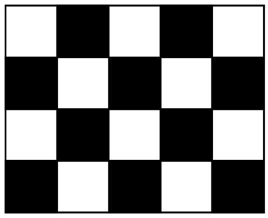
But

$$1 + 1 + 1 = 3 \qquad 1 + 1 + 7 = 9 \qquad 1 + 7 + 7 = 15 \qquad 7 + 7 + 7 = 21$$

which are all divisible by 3.

Hence there are no 3-digit numbers *N* satisfying both these conditions.

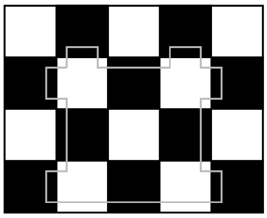
B6 On the 4 × 5 grid shown, I am only allowed to move from one square to a neighbouring square by crossing an edge. So the squares I visit alternate between black and white. I have to start on a black square and visit each black square exactly once. What is the smallest number of white squares that I have to visit? Prove that your answer is indeed the smallest.



(If I visit a white square more than once, I only count it as one white square visited).

Solution

It is possible to visit each black square exactly once by travelling through 4 white squares (as shown in this diagram).



Suppose there is a route using only three white squares. The maximum number of black squares adjacent to the first white square on the route is 4. To reach the second white square on the route, the route must pass via one of those black squares − and so there are no more than 3 additional black squares adjacent to the second white square. Likewise, when the third white square is reached there are at most 3 additional black squares adjacent to it. This means that with 3 white squares we can reach at most 10 black squares and, moreover, we can only reach 10 if each of the three white squares is adjacent to 4 black squares. However, in the given diagram, there are only three such white squares − and none of them is adjacent to the black squares in the bottom corners. Hence three white squares is not enough. This means that the smallest number of white squares I have to visit is four.