STEP Support Programme

Hints and Partial Solutions for Assignment 8

Warm-up

1 **When you are asked to prove something it is best not to start with what you are trying to prove**. The question states "by considering $(x-y)^2$" so the first line of your proof should be $(x-y)^2 \geqslant 0$.

Working backwards can be ok if the implication works both ways each time (so that you can put a $\iff$ in between each line of working). If you are working backwards then you do need to put the $\iff$ signs in, for example:

$$\begin{aligned} & x^2+y^2 \geqslant 2xy \\ \iff\ & x^2+y^2-2xy \geqslant 0 \\ \iff\ & (x-y)^2 \geqslant 0 \end{aligned}$$

However, this is not as elegant as "doing it the right way around", i.e. making your first statement $(x-y)^2 \geqslant 0$.

Equality holds when $(x-y)^2 = 0$ (i.e. when $x = y$).
Once you have shown that $x^2+y^2 \geqslant 2xy$ you can use the substitution $a = x^2$ and $b = y^2$, which is allowed as both a and b are non-negative, to get the AM-GM result (you don't have to start all over again).

(i) Again, you should not start with the result you are trying to prove. Multiplying by 2 should give you a hint for how you might approach this and hopefully suggests that could start by considering $(x-y)^2+(y-z)^2+(y-x)^2 \geqslant 0$.

(ii) Here you use AM-GM (the starred equation in introduction to the question) three times: twice to obtain the first inequality then once to obtain the second inequality. The last application of the inequality uses $a = \sqrt{pq}$ and $b = \sqrt{rs}$.

(iii) This was rather tricky. The first part is not so bad — use the result in part **(ii)** with $s = \dfrac{p+q+r}{3}$.

For the second part ('Deduce ...') start by writing the inequality as

$$\frac{p+q+r}{3} \geqslant \sqrt[4]{pqr} \times \sqrt[4]{\frac{p+q+r}{3}}$$

and then divide both sides by $\sqrt[4]{\dfrac{p+q+r}{3}}$ before raising to the power 4 and then taking the cube root. Alternatively, you could raise both parts to the power 4, before dividing by $\dfrac{p+q+r}{3}$ and then taking the cube root.

Preparation

2 **(i)** $(x+2)^2 + (y-9)^2 = 25$.

(ii) The first case has two intersections between the circles, at $(\pm 3, 4)$. The second has just one intersection at $(5, 0)$ and the third has no intersections (you should find that $y = \frac{73}{14}$ which is greater than 5).

(iii) You can do this by pure geometry — the two points of intersection are symmetric about the y-axis and since the centre of the circle must lie on the perpendicular bisector of any chord the centre must be on the y-axis.

You can instead use coordinate geometry. For this second approach substitute each of the two pairs of coordinates of the points of intersection (i.e. $x = \pm 3$ and $y = 4$) into $(x-a)^2 + (y-b)^2 = r^2$ and use the two resulting equations to show that $a = 0$.

If the centre is at $(0, 2)$ and the circle passes through $(3, 4)$ then $x^2 + (y-2)^2 = 13$.

(iv) Here we have a circle and ellipse (you don't have to know anything about ellipses) meeting in three different places, $(\pm 2, 0)$ and $(0, -2)$. At one of the intersections the two curves are touching, i.e. their gradients are the same at this point (for the other two points the curves are crossing).

Try using Desmos to sketch the curves.

The STEP question

3 Eliminate y first and then solve the resulting quadratic equation in x. You should find that one of your values of x is possible, but the other (the larger one) gives imaginary values for y. You can use the position of the points of intersection to show that the centre of the circle lies on the x axis, so at $(a, 0)$ say (and you can do this algebraically or by using a geometrical argument). You can then write the equation of the circle as $(x-a)^2 + y^2 = r^2$ and use the distance from $(a, 0)$ to one of your points of intersection to write r^2 in terms of a. Since the answer is given, I won't write it down again!

If the answer is given them you **must** ensure that there are no gaps in your argument.

Warm down

4 **(i)** 3 socks might not be enough (as you can have 1 red, 1 blue and 1 green). But then as the next sock you take must be red, blue or green you will have a matching pair if you take 4 socks.

(ii) As before, with 3 socks you can have 1 red, 1 blue and 1 green. Suppose that the next sock is red (it doesn't matter what colour it is) so with 4 socks you now have 1 pair of (red) socks. If the fifth sock is blue or green you now have 2 pairs and are finished, however the fifth sock may be red again so 5 socks are not enough to ensure that you have two pairs. In this case, the sixth sock will be either red, green or blue and hence match up with one of the other socks.

(iii) You could argue as follows, perhaps putting in a little more detail.

Suppose you have $2n$ socks. Since $2n$ is even, there are two possibilities:

(1) either you have an even number of each colour;

(2) or you have an even number of one colour (green, say) and an odd number of red socks and odd number of blue socks.

Why can you not have an odd number of each colour?

In case **(1)**, you have n pairs, so you are done.

In case **(2)**, you have $n-1$ pairs and two left-over socks, one red and one blue. Now if you pick one more sock, it may be green, in which case you still wouldn't have n pairs. So you must pick two more and this gives n pairs whatever colour they are.