STEP Support Programme

Hints and Partial Solutions for Assignment 10

Warm-up

1 **(i)** Note that $AP = \tan\alpha$ and $PC = \tan\beta$. You can also show that $AB = \frac{1}{\cos\alpha}$. The result $\sin(90° - \beta) = \cos\beta$ will be helpful.

Do be careful with brackets, write

$$\cos\alpha\cos\beta \times (\tan\alpha + \tan\beta)$$

rather than

$$\cos\alpha\cos\beta \times \tan\alpha + \tan\beta.$$

You can find $\sin(\alpha - \beta)$ from $\sin(\alpha + (-\beta))$ using the previous result, along with $\sin(-\beta) = -\sin\beta$ and $\cos(-\beta) = \cos\beta$.

You can find $\cos(\alpha+\beta)$ from $\cos(\alpha+\beta) = \sin(90° - (\alpha+\beta)) = \sin((90° - \alpha) - \beta)$ and similarly for $\cos(\alpha - \beta)$.

Note that when you have the expression for $\sin(\alpha + \beta)$, you can derive $\sin(\alpha - \beta)$ and $\cos(\alpha \pm \beta)$ with hardly any extra work. With these you can also derive $\tan(\alpha \pm \beta)$ fairly painlessly.

(ii) $\sin 75° = \sin(45° + 30°)$ and $\sin 15° = \sin(45° - 30°)$. At the risk of repeating ourselves: when a question asks for **values** then decimal approximations are not what are wanted.

Answers: $\sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(iii) You will need expressions for $\sin 2A$ and $\cos 2A$, but these can be found from the results in part **(i)**, for example $\sin 2A = \sin(A + A) = \cdots$. You will probably also need to use $\sin^2 A + \cos^2 A = 1$.

It is a good idea to check your answer, for example by substituting $A = 30°$.

Answer: $\cos 3A = 4\cos^3 A - 3\cos A$.

Preparation

2 **(i)** Things will be much easier if you write $1-\frac{1}{2}=\frac{1}{2}$ etc. before you try to multiply. The prime factors of 120 or 2, 3, and 5. The final answer is 32.

(ii) Note that if a is a *positive* integer then $a-1$ is an integer satisfying $a-1\geqslant 0$. If you expand you get $x^a - x^{a-1}$, and since x and a are integers and $a-1\geqslant 0$ this is an integer subtracted from an integer, hence is an integer!

(iii) $39600 = 2^4\times 3^2\times 5^2\times 11$ and $52920 = 2^3\times 3^3\times 5\times 7^2$ so the HCF is $2^3\times 3^2\times 5 = 360$.

(iv) These sorts of statements can be a bit confusing. I usually re-order them in my head, so

"$a=b$ **if** $a^2=b^2$" becomes "**if** $a^2=b^2$ **then** $a=b$" which I can now see is false

and

"$a=b$ **only if** $a^2=b^2$" becomes "**only if** $a^2=b^2$ **is it the case that** $a=b$" which I can now see is true.

(a) True: ab even $\Leftarrow$ a and b both even.
(b) False as ab would be even if just one of a and b were even, so ab even $\not\Rightarrow$ a and b both even.
(c) False: either $a=b$ or $a=-b$ if $a^2=b^2$, so $a=b\not\Leftarrow a^2=b^2$.
(d) True: $a=b\Rightarrow a^2=b^2$.
(e) True: equilateral $\Leftarrow$ three equal sides.
(f) True: equilateral $\Rightarrow$ three equal sides.

(v) **(a)** True.
(b) False. An odd number is prime **if** it is three, but this is not an **only if**.
(c) False. $x=3$ only if $x^2-9=0$ (i.e. this is not an **if**).
(d) False. It is certainly the case that the triangle is right-angled **if** $a^2+b^2=c^2$, but this is not a necessary condition (it is not **only if**): the triangle is also right-angled if $a^2=b^2+c^2$ (i.e. a is the hypotenuse).
(e) True.

The STEP question

3 **(i)** **(a)** $f(12) = 4$ and $f(180) = 48$.

(b) If you use the hint provided and write $N = p_1^{a_1} \times p_2^{a_2} \times \cdots$ then you should find that $f(N) = p_1^{a_1-1}(p_1-1)p_2^{a_2-1}(p_2-1)\cdots$. You can then explain why this is an integer.

(ii) **(a)** A simple counterexample (remember the simpler the better!) would be $f(12) \neq f(2) \times f(6)$. You show it is indeed a counterexample by calculating the three values of the function.

(b) This result is true, and can be shown just by using the definitions such as $f(p) = p\left(1 - \frac{1}{p}\right)$. When p and q are not distinct ($p = q$), the result is not true.

(c) Try $f(4)$ and $f(15)$. The statement $f(p)f(q) = f(pq)$ holds as long as p and q are *co-prime*, that is that they have no prime factors in common.
When p and q have a prime factor in common then $f(pq) \neq f(p)f(q)$

(iii) Start by writing $f(p^m) = p^m\left(1 - \frac{1}{p}\right) = p^{m-1}(p-1)$. Then express 146410 as a product and compare this to $p^{m-1}(p-1)$ so that you get $p^{m-1}(p-1) = 11^4 \times 10$. You should then be able to spot what p and m are.

Warm down

4 **(i)** Leave answers in fractions (unless specifically asked for a decimal approximation). Answer: $\frac{144}{13}$.

(ii) There is nothing wrong with the negative solution (nothing in the question says that the numbers have to be positive) so put down both solutions. Answer: $(7, 4)$ or $(-4, -7)$.

(iii) Two of the terms simplify nicely, but the other two do not.
Answer: $ax + by - a^{\frac{1}{4}}b^{\frac{2}{5}}x^{\frac{1}{3}}y^{\frac{1}{6}} - a^{\frac{3}{4}}b^{\frac{3}{5}}x^{\frac{2}{3}}y^{\frac{5}{6}}$.

(iv) Start by multiplying numerator and denominator by $\sqrt{3} - 1$.

Simplifying $\sqrt{12 + 6\sqrt{3}} \times (\sqrt{3} - 1)$ can be little complicated. One way to proceed it to take the $\sqrt{3} - 1$ inside the square root, so simplify $\sqrt{(12 + 6\sqrt{3}) \times (\sqrt{3} - 1)^2}$. Everything should simplify nicely to give $\sqrt{3}$.