

STEP Support Programme

Hints and Partial Solutions for Assignment 10

Warm-up

- 1 (i) Note that $AP = \tan \alpha$ and $PC = \tan \beta$. You can also show that $AB = \frac{1}{\cos \alpha}$. The result $\sin(90^\circ - \beta) = \cos \beta$ will be helpful.

Do be careful with brackets, write

$$\cos \alpha \cos \beta \times (\tan \alpha + \tan \beta)$$

rather than

$$\cos \alpha \cos \beta \times \tan \alpha + \tan \beta.$$

You can find $\sin(\alpha - \beta)$ from $\sin(\alpha + (-\beta))$ using the previous result, along with $\sin(-\beta) = -\sin \beta$ and $\cos(-\beta) = \cos \beta$.

You can find $\cos(\alpha + \beta)$ from $\cos(\alpha + \beta) = \sin(90^\circ - (\alpha + \beta)) = \sin((90^\circ - \alpha) - \beta)$ and similarly for $\cos(\alpha - \beta)$.

Note that when you have the expression for $\sin(\alpha + \beta)$, you can derive $\sin(\alpha - \beta)$ and $\cos(\alpha \pm \beta)$ with hardly any extra work. With these you can also derive $\tan(\alpha \pm \beta)$ fairly painlessly.

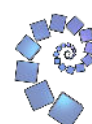
- (ii) $\sin 75^\circ = \sin(45^\circ + 30^\circ)$ and $\sin 15^\circ = \sin(45^\circ - 30^\circ)$. At the risk of repeating ourselves: when a question asks for **values** then decimal approximations are not what are wanted.

$$\text{Answers: } \sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}} \text{ and } \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

- (iii) You will need expressions for $\sin 2A$ and $\cos 2A$, but these can be found from the results in part (i), for example $\sin 2A = \sin(A + A) = \dots$. You will probably also need to use $\sin^2 A + \cos^2 A = 1$.

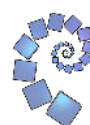
It is a good idea to check your answer, for example by substituting $A = 30^\circ$.

$$\text{Answer: } \cos 3A = 4 \cos^3 A - 3 \cos A.$$



Preparation

- 2 (i) Things will be much easier if you write $1 - \frac{1}{2} = \frac{1}{2}$ etc. before you try to multiply. The prime factors of 120 are 2, 3, and 5. The final answer is 32.
- (ii) Note that if a is a *positive* integer then $a - 1$ is an integer satisfying $a - 1 \geq 0$. If you expand you get $x^a - x^{a-1}$, and since x and a are integers and $a - 1 \geq 0$ this is an integer subtracted from an integer, hence is an integer!
- (iii) $39600 = 2^4 \times 3^2 \times 5^2 \times 11$ and $52920 = 2^3 \times 3^3 \times 5 \times 7^2$ so the HCF is $2^3 \times 3^2 \times 5 = 360$.
- (iv) These sorts of statements can be a bit confusing. I usually re-order them in my head, so
 “ $a = b$ **if** $a^2 = b^2$ ” becomes “**if** $a^2 = b^2$ **then** $a = b$ ” which I can now see is false
 and
 “ $a = b$ **only if** $a^2 = b^2$ ” becomes “**only if** $a^2 = b^2$ **is it the case that** $a = b$ ” which I can now see is true.
- (a) True: ab even $\Leftrightarrow a$ and b both even.
 (b) False as ab would be even if just one of a and b were even, so ab even $\nRightarrow a$ and b both even.
 (c) False: either $a = b$ or $a = -b$ if $a^2 = b^2$, so $a = b \nRightarrow a^2 = b^2$.
 (d) True: $a = b \Rightarrow a^2 = b^2$.
 (e) True: equilateral \Leftrightarrow three equal sides.
 (f) True: equilateral \Rightarrow three equal sides.
- (v) (a) True.
 (b) False. An odd number is prime **if** it is three, but this is not an **only if**.
 (c) False. $x = 3$ only if $x^2 - 9 = 0$ (i.e. this is not an **if**).
 (d) False. It is certainly the case that the triangle is right-angled **if** $a^2 + b^2 = c^2$, but this is not a necessary condition (it is not **only if**): the triangle is also right-angled if $a^2 = b^2 + c^2$ (i.e. a is the hypotenuse).
 (e) True.



The STEP question

- 3 (i) (a) $f(12) = 4$ and $f(180) = 48$.
 (b) If you use the hint provided and write $N = p_1^{a_1} \times p_2^{a_2} \times \dots$ then you should find that $f(N) = p_1^{a_1-1}(p_1 - 1)p_2^{a_2-1}(p_2 - 1) \dots$. You can then explain why this is an integer.
- (ii) (a) A simple counterexample (remember the simpler the better!) would be $f(12) \neq f(2) \times f(6)$. You show it is indeed a counterexample by calculating the three values of the function.
 (b) This result is true, and can be shown just by using the definitions such as $f(p) = p \left(1 - \frac{1}{p}\right)$. When p and q are not distinct ($p = q$), the result is not true.
 (c) Try $f(4)$ and $f(15)$. The statement $f(p)f(q) = f(pq)$ holds as long as p and q are *co-prime*, that is that they have no prime factors in common.
 When p and q have a prime factor in common then $f(pq) \neq f(p)f(q)$
- (iii) Start by writing $f(p^m) = p^m \left(1 - \frac{1}{p}\right) = p^{m-1}(p - 1)$. Then express 146410 as a product and compare this to $p^{m-1}(p - 1)$ so that you get $p^{m-1}(p - 1) = 11^4 \times 10$. You should then be able to spot what p and m are.

Warm down

- 4 (i) Leave answers in fractions (unless specifically asked for a decimal approximation).
 Answer: $\frac{144}{13}$.
- (ii) There is nothing wrong with the negative solution (nothing in the question says that the numbers have to be positive) so put down both solutions. Answer: $(7, 4)$ or $(-4, -7)$.
- (iii) Two of the terms simplify nicely, but the other two do not.
 Answer: $ax + by - a^{\frac{1}{4}}b^{\frac{2}{5}}x^{\frac{1}{3}}y^{\frac{1}{6}} - a^{\frac{3}{4}}b^{\frac{3}{5}}x^{\frac{2}{3}}y^{\frac{5}{6}}$.
- (iv) Start by multiplying numerator and denominator by $\sqrt{3} - 1$.
 Simplifying $\sqrt{12 + 6\sqrt{3}} \times (\sqrt{3} - 1)$ can be little complicated. One way to proceed it to take the $\sqrt{3} - 1$ inside the square root, so simplify $\sqrt{(12 + 6\sqrt{3}) \times (\sqrt{3} - 1)^2}$. Everything should simplify nicely to give $\sqrt{3}$.

