

STEP Support Programme

Hints and partial solutions for Assignment 25

Warm-up

- 1 (i) (a) The integral becomes:

$$\begin{aligned}\int \frac{u+2}{u} du &= \int 1 + \frac{2}{u} du \\ &= u + 2 \ln |u| + c \\ &= x - 2 + 2 \ln |x - 2| + c.\end{aligned}$$

You can, if you wish, combine the “−2” with the constant of integration to get a final answer of $x + 2 \ln |x - 2| + k$. This is not a necessary step, but does look a little nicer.

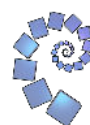
- (b) Here we have:

$$\begin{aligned}\int \frac{3(u^2 - 1)}{u} \times u du &= 3 \int (u^2 - 1) du \\ &= 3 \left(\frac{1}{3} u^3 - u \right) + c \\ &= u^3 - 3u + c \\ &= (2x + 1)^{\frac{3}{2}} - 3(2x + 1)^{\frac{1}{2}} + c\end{aligned}$$

The final answer could also be written as $2(x - 1)\sqrt{2x + 1} + c$.

- (ii) In the case the integral becomes:

$$\begin{aligned}\int_0^{\frac{1}{6}\pi} \frac{1}{\sqrt{4 - 4 \sin^2 \theta}} \times 2 \cos \theta d\theta &= \int_0^{\frac{1}{6}\pi} \frac{1}{2 \cos \theta} \times 2 \cos \theta d\theta \\ &= \int_0^{\frac{1}{6}\pi} 1 d\theta \\ &= \left[\theta \right]_0^{\frac{1}{6}\pi} \\ &= \frac{1}{6}\pi\end{aligned}$$



Preparation

- 2 (i) Since you are given A and B in the question you should stick with these variables (rather than using a and b). We have:

$$\begin{aligned}\tan(A - B) &= \frac{\sin(A - B)}{\cos(A - B)} \\ &= \frac{\sin A \cos B - \sin B \cos A}{\cos A \cos B + \sin A \sin B} \\ &= \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\sin B \cos A}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}} \\ &= \frac{\tan A - \tan B}{1 + \tan A \tan B}\end{aligned}$$

- (ii) One approach is:

$$\begin{aligned}\ln\left(1 + \frac{\frac{1}{2} - x}{\frac{1}{2} + x}\right) &= \ln\left(\frac{\frac{1}{2} + x + \frac{1}{2} - x}{\frac{1}{2} + x}\right) \\ &= \ln\left(\frac{1}{\frac{1}{2} + x}\right) \\ &= \ln 1 - \ln\left(\frac{1}{2} + x\right) \\ &= -\ln\left(\frac{1}{2} + x\right)\end{aligned}$$

- (iii) Dividing throughout by $\cos^2 \theta$ gives $\tan^2 \theta + 1 \equiv \sec^2 \theta$.

- (iv) The quotient rule (using a prime to denote differentiation)

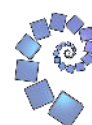
$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

follows immediately from the product rule applied to uv^{-1} using the chain rule to differentiate v^{-1} (which gives $-v^{-2}v'$).

Using this we have:

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{\sin \theta}{\cos \theta}\right) &= \frac{\cos \theta \times \cos \theta - (-\sin \theta) \times \sin \theta}{\cos^2 \theta} \\ &= \frac{1}{\cos^2 \theta} \\ &= \sec^2 \theta.\end{aligned}$$

Another result which is good to “know by heart”!



(v) One approach is:

$$\begin{aligned}\frac{1 + \sin 2\alpha}{1 + \cos 2\alpha} &= \frac{\cos^2 \alpha + \sin^2 \alpha + 2 \sin \alpha \cos \alpha}{\cos^2 \alpha + \sin^2 \alpha + \cos^2 \alpha - \sin^2 \alpha} \\ &= \frac{(\sin \alpha + \cos \alpha)^2}{2 \cos^2 \alpha} \\ &= \frac{1}{2} \left(\frac{\sin \alpha + \cos \alpha}{\cos \alpha} \right)^2 \\ &= \frac{1}{2} (\tan \alpha + 1)^2\end{aligned}$$

(vi) The answer is given, so you do need to show some working:

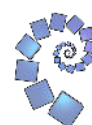
$$\begin{aligned}\int_0^{84} \frac{x^2}{x^2 + (84 - x)^2} dx &= \int_{84}^0 \frac{(84 - u)^2}{(84 - u)^2 + u^2} \times -1 du \\ &= -1 \times - \int_0^{84} \frac{(84 - u)^2}{u^2 + (84 - u)^2} du \\ &= \int_0^{84} \frac{(84 - x)^2}{x^2 + (84 - x)^2} dx\end{aligned}$$

where the last step involves a substitution of $u = x$.

We then have:

$$\begin{aligned}I + I &= \int_0^{84} \frac{x^2}{x^2 + (84 - x)^2} dx + \int_0^{84} \frac{(84 - x)^2}{x^2 + (84 - x)^2} dx \\ &= \int_0^{84} \frac{x^2 + (84 - x)^2}{x^2 + (84 - x)^2} dx \\ &= \int_0^{84} 1 dx = 84\end{aligned}$$

so $2I = 84$ and hence $I = 42$.



The STEP question

3 Using the given substitution we have:

$$\begin{aligned}
 I &= \int_0^{\frac{1}{4}\pi} \ln(1 + \tan \theta) \, d\theta = \int_{\frac{1}{4}\pi}^0 \ln(1 + \tan(\tfrac{1}{4}\pi - \phi)) \times -1 \, d\phi \\
 &= \int_0^{\frac{1}{4}\pi} \ln\left(1 + \frac{1 - \tan \phi}{1 + \tan \phi}\right) \, d\phi \quad \text{using } \tan(\tfrac{1}{4}\pi) = 1 \\
 &= \int_0^{\frac{1}{4}\pi} \ln\left(\frac{1 + \cancel{\tan \phi} + 1 - \cancel{\tan \phi}}{1 + \tan \phi}\right) \, d\phi \\
 &= \int_0^{\frac{1}{4}\pi} (\ln 2 - \ln(1 + \tan \phi)) \, d\phi \\
 &= \int_0^{\frac{1}{4}\pi} \ln 2 \, d\phi - I \\
 &= \left[\phi \ln 2\right]_0^{\frac{1}{4}\pi} - I \\
 &= \tfrac{1}{4}\pi \ln 2.
 \end{aligned}$$

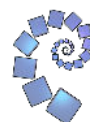
Don't forget that $\ln 2$ is a constant so when we integrate we get $\phi \times \ln 2$. We now have $I = \frac{1}{4}\pi \ln 2 - I$ and so we have $I = \frac{1}{8}\pi \ln 2$.

- (i) Looking at the limits, and thinking that we would like something that looks a little like the first integral, try $x = \tan \theta$.

$$\begin{aligned}
 \int_0^1 \frac{\ln(1+x)}{1+x^2} \, dx &= \int_0^{\frac{1}{4}\pi} \frac{\ln(1+\tan \theta)}{\cancel{1+\tan^2 \theta}} \times \sec^2 \theta \, d\theta \\
 &= \tfrac{1}{8}\pi \ln 2.
 \end{aligned}$$

- (ii) You can get a hint of what to use by looking at the upper limit of the integral. Starting with $x = 2u$ gives:

$$\begin{aligned}
 \int_0^{\frac{1}{2}\pi} \ln\left(\frac{1+\sin x}{1+\cos x}\right) \, dx &= \int_0^{\frac{1}{4}\pi} \ln\left(\frac{1+\sin 2u}{1+\cos 2u}\right) \times 2 \, du \\
 &= \int_0^{\frac{1}{4}\pi} \ln\left(\tfrac{1}{2}(1+\tan u)^2\right) \times 2 \, du \quad \text{using 2(v)} \\
 &= 2 \int_0^{\frac{1}{4}\pi} \ln\left(\tfrac{1}{2}\right) \, du + 4 \int_0^{\frac{1}{4}\pi} \ln(1+\tan u) \, du \\
 &= 2\left[u \times \ln\left(\tfrac{1}{2}\right)\right]_0^{\frac{1}{4}\pi} + 4 \times \tfrac{1}{8}\pi \ln 2 \\
 &= 2 \times \tfrac{1}{4}\pi \times (-\ln 2) + \tfrac{1}{2}\pi \ln 2 = 0
 \end{aligned}$$



If you hadn't done the preparation questions first, you will have had to do some more lines of working to get to the expression in $\tan u$. However, knowing what sort of expression you are aiming for is a big help, and using the limits to suggest a suitable substitution is usually a good idea.

Other methods are also possible!

Warm down

- 4 (i) Use $t = a - x$ to give:

$$\begin{aligned} I &= \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx = \int_a^0 \frac{f(a-t)}{f(a-t) + f(t)} \times -1 dt \\ &= \int_0^a \frac{f(a-t)}{f(a-t) + f(t)} dt \\ &= \int_0^a \frac{f(a-x)}{f(a-x) + f(x)} dx \end{aligned}$$

Then we have $2I = \int_0^a 1 dx = a$ and hence $I = \frac{1}{2}a$.

For the last part, we can use $\sin x = \cos(\frac{1}{2}\pi - x)$ to get:

$$\int_0^{\frac{1}{2}\pi} \frac{\cos x}{\cos x + \cos(\frac{1}{2}\pi - x)} dx$$

which is now in the same form, so the integral is equal to $\frac{1}{2}a = \frac{1}{4}\pi$.

- (ii) The result $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ is well worth memorising!

(a) Here $f(x) = \sin x$, so the integral is $\left[\ln|\sin(x)| \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \ln 1 - \ln \frac{1}{\sqrt{2}} = \frac{1}{2} \ln 2$.

(b) Writing the integral in the form $\int \frac{\frac{1}{x}}{\ln x} dx$ gives the result

$$\int \frac{1}{x \ln x} dx = \ln|\ln x| + c.$$

(c) $S + T = x + c_1$ and $S - T = \ln(\cos x + \sin x) + c_2$. You can then solve these to give:

$$\begin{aligned} S &= \frac{1}{2}x + \frac{1}{2} \ln(\cos x + \sin x) + k_1 \\ T &= \frac{1}{2}x - \frac{1}{2} \ln(\cos x + \sin x) + k_2. \end{aligned}$$

