UK Junior Mathematical Olympiad 2014

Organised by The United Kingdom Mathematics Trust

Thursday 12th June 2014

RULES AND GUIDELINES : READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, measuring instruments and squared paper is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work. Write in blue or black pen or pencil.

   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 30 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you are not able to do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like π, fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

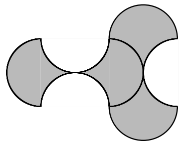
Section A

A1. What is the largest digit that appears in the answer to the calculation $(3 \times 37)^2$?

A2. What is the sum of all fractions of the form $\frac{N}{7}$, where N is a positive integer less than 7?

A3. The six angles of two different triangles are listed in decreasing order. The list starts 115°, 85°, 75° and 35°. What is the last angle in the list?

A4. The figure shows two shapes that fit together exactly. Each shape is formed by four semicircles of radius 1. What is the total shaded area?



A5. The integer 113 is prime, and its 'reverse' 311 is also prime. How many two-digit primes are there between 10 and 99 which have the same property?

A6. A square of side length 1 is drawn. A larger square is drawn around it such that all parallel sides are a distance 1 apart. This process continues until the total perimeter of the squares drawn is 144.

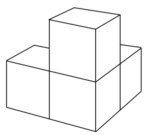
What is the area of the largest square drawn?

A7. The time is 20:14. What is the smaller angle between the hour hand and the minute hand on an accurate analogue clock?

A8. Sam has four cubes all the same size: one blue, one red, one white and one yellow. She wants to glue the four cubes together to make the solid shape shown.

How many differently-coloured shapes can Sam make?

[Two shapes are considered to be the same if one can be picked up and turned around so that it looks identical to the other.]



A9. A rectangle is made by placing together three smaller rectangles P, Q and R, without gaps or overlaps. Rectangle P measures $3\text{ cm} \times 8\text{ cm}$ and Q measures $2\text{ cm} \times 5\text{ cm}$.

How many possibilities are there for the measurements of R?

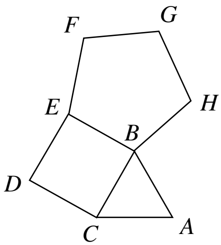
A10. My four pet monkeys and I harvested a large pile of peanuts. Monkey A woke in the night and ate half of them; then Monkey B woke and ate one third of what remained; then Monkey C woke and ate one quarter of the rest; finally Monkey D ate one fifth of the much diminished remaining pile. What fraction of the original harvest was left in the morning?

Section B

Your solutions to Section B will have a major effect on your JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

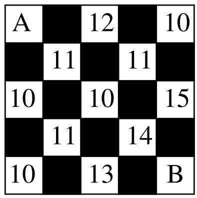
B1. The figure shows an equilateral triangle *ABC*, a square *BCDE*, and a regular pentagon *BEFGH*.

What is the difference between the sizes of $\angle ADE$ and $\angle AHE$?



B2. I start at the square marked A and make a succession of moves to the square marked B. Each move may only be made downward or to the right. I take the sum of all the numbers in my path and add 5 for every black square I pass through.

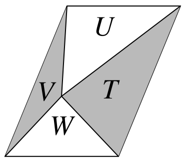
How many paths give a sum of 51?



B3. A point lying somewhere inside a parallelogram is joined to the four vertices, thus creating four triangles T, U, V and W, as shown.

Prove that

$$\text{area } T + \text{area } V = \text{area } U + \text{area } W.$$



B4. There are 20 sweets on the table. Two players take turns to eat as many sweets as they choose, but they must eat at least one, and never more than half of what remains. The loser is the player who has no valid move.

Is it possible for one of the two players to force the other to lose? If so, how?

B5. Find a fraction $\frac{m}{n}$, with m not equal to n, such that all of the fractions

$$\frac{m}{n}, \frac{m+1}{n+1}, \frac{m+2}{n+2}, \frac{m+3}{n+3}, \frac{m+4}{n+4}, \frac{m+5}{n+5}$$

can be simplified by cancelling.

B6. The sum of four different prime numbers is a prime number. The sum of some pair of the numbers is a prime number, as is the sum of some triple of the numbers. What is the smallest possible sum of the four prime numbers?

UK Junior Mathematical Olympiad 2014 Solutions

A1 **3** Firstly, $3 \times 37 = 111$ and so $(3 \times 37)^2 = 111^2$. Now

$$111^2 = 1 \times 111 + 10 \times 111 + 100 \times 111$$
$$= 111 + 1110 + 11100$$
$$= 12\,321.$$

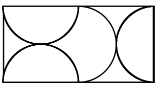
Therefore the largest digit is 3.

A2 **3** The sum in question is

$$\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{5}{7} + \frac{6}{7} = \frac{21}{7} = 3.$$

A3 **20°** First note that $115° + 85° > 180°$ and $115° + 75° > 180°$ so one triangle contains both the 75° and the 85° angles. Also note that $85° + 75° + 35° > 180°$ so that triangle does not contain the 35° angle. Hence one triangle must have internal angles including 85° and 75°, and the other triangle must have internal angles 115° and 35°. The two remaining angles are therefore $180° - (115° + 35°) = 30°$ and $180° - (85° + 75°) = 20°$. Therefore the last angle in the list is 20°.

A4 **8** The shapes can be cut and rearranged to make a 4×2 rectangle as shown.



Therefore the shaded area is 8.

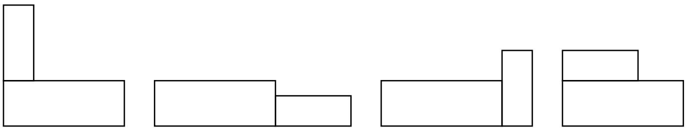
A5 **9** Any number ending in 2, 4, 6 or 8 is even. Similarly, any number ending in 5 is divisible by 5. Hence, for both a two-digit number and its reverse to be a prime, the original number can only start with 1, 3, 7 or 9. There are 10 two-digit primes starting with 1, 3, 7 or 9, namely 11, 13, 17, 19, 31, 37, 71, 73, 79 and 97 and, of these, only 19 does not have its reverse in the list. Hence there are 9 two-digit primes with the desired property.

A6 **121** The squares have side lengths 1, 3, 5, 7, 9, 11, . . . and so the sums of the perimeters are 4, 16, 36, 64, 100, 144, ... Thus the largest square has side-length 11 and area 121.

A7 **163°** The minute hand takes 60 minutes to make a complete turn, and so rotates through $360° \div 60 = 6°$ in one minute. Therefore, at 14 minutes past the hour, the minute hand has rotated by $14 \times 6° = 84°$. The hour hand takes 12 hours, or 720 minutes, to make a complete turn and so rotates through 0.5° in one minute. Therefore, at 20:14, the hour hand has rotated through $240° + 7° = 247°$. Thus the angle between the minute hand and the hour hand is $247° - 84° = 163°$.

A8 **8** The 'corner' cube may be chosen in four ways. Given a choice of the 'corner' cube, there are then three choices for the top cube and a further two choices for the left-hand cube. This gives $4 \times 3 \times 2 = 24$ different ways of arranging the cubes. However, the shape can be rotated so that each of the three faces of the 'corner' cube that are not joined to any other cube are at the bottom and the shape would then look the same. So the set of 24 arrangements contains groups of three that can be rotated into each other. Hence the number of differently coloured shapes is $24 \div 3 = 8$.

A9 **4** The rectangles P and Q must be placed together edge-to-edge in one of the following ways.

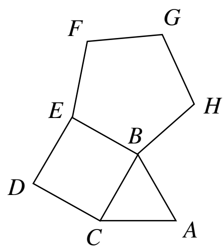


Therefore there are 4 possibilities for the measurements of R: 6×5, 1×5, 8×2 and 3×2.

A10 $\frac{1}{5}$ After Monkey A has eaten half of the pile, the fraction of the original pile that remains is $\frac{1}{2}$. Monkey B eats $\frac{1}{3}$ of the remaining pile and so leaves $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ of the original pile. Monkey C leaves $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$; and Monkey D leaves $\frac{4}{5} \times \frac{1}{4} = \frac{1}{5}$ of the original pile.

B1 The figure shows an equilateral triangle *ABC*, a square *BCDE*, and a regular pentagon *BEFGH*.

What is the difference between the sizes of $\angle ADE$ and $\angle AHE$?



Solution

We calculate the sizes of $\angle ADE$ and $\angle AHE$ in turn. Since *ABC* is an equilateral triangle, $\angle ACB = 60°$. Since *BCDE* is a square, $\angle BCD = 90°$. As edge *BC* is shared by the triangle and the square, $AC = CD$. Therefore the triangle *ACD* is isosceles. Now $\angle ACD = 60° + 90° = 150°$ and so $\angle ADC = 15°$. Therefore

$\angle ADE = \angle EDC - \angle ADC = 90° - 15° = 75°$.

Now, for angle $\angle AHE$, $\angle EBH = 108°$ as *BEFGH* is a regular pentagon. By considering the angles around *B*, $\angle ABH = 360° - (108° + 90° + 60°) = 102°$. Since triangle *ABH* is isosceles, this means that $\angle AHB = 39°$. Also, triangle *HBE* is isosceles and so $\angle BHE = 36°$. Therefore $\angle AHE = \angle AHB + \angle BHE = 39° + 36° = 75°$.

So the difference between the sizes of the angles is zero.

B2 I start at the square marked A and make a succession of moves to the square marked B. Each move may only be made downward or to the right. I take the sum of all the numbers in my path and add 5 for every black square I pass through.

How many paths give a sum of 51?

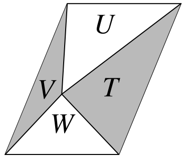
A		12		10
	11		11	
10		10		15
	11		14	
10		13		B

Solution

Any path from A to B must pass through four black squares, contributing 20 to the sum.

To have a path with sum 51, the numbers in the remaining three squares must sum to 31. Since all the numbers in the squares have two digits, the only possible way to make a sum of 31 is $10 + 10 + 11$. However any path must pass through the diagonal containing the numbers 13, 14 and 15. Thus there are no paths giving a sum of 51.

B3 A point lying somewhere inside a parallelogram is joined to the four vertices, thus creating four triangles T, U, V and W, as shown.

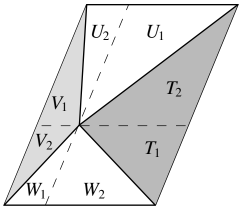


Prove that

$$\text{area } T + \text{area } V = \text{area } U + \text{area } W.$$

Solution

The parallelogram may also be split into four parallelograms, each having the point as a vertex.



If we label the separate triangles formed as shown in the diagram then it can be seen that area V_1 = area U_2, area U_1 = area T_2, area T_1 = area W_2 and area W_1 = area V_2.
Therefore

$$\begin{aligned}\text{area } T + \text{area } V &= \text{area } T_1 + \text{area } T_2 + \text{area } V_1 + \text{area } V_2 \\ &= \text{area } W_2 + \text{area } U_1 + \text{area } U_2 + \text{area } W_1 \\ &= \text{area } U_1 + \text{area } U_2 + \text{area } W_1 + \text{area } W_2 \\ &= \text{area } U + \text{area } W.\end{aligned}$$

B4 There are 20 sweets on the table. Two players take turns to eat as many sweets as they choose, but they must eat at least one, and never more than half of what remains. The loser is the player who has no valid move.

Is it possible for one of the two players to force the other to lose? If so, how?

Solution

The losing player is the one who is left with 1 sweet on the table, because taking that sweet would mean taking more than half of what remains. The first player can force the second to lose by leaving 15, 7, 3 and 1 sweets after successive turns. Call the first player A and the second player B. On her first turn, A should leave 15 sweets. Then B must leave between 8 and 14 sweets (inclusive). No matter how many sweets are left, A should leave 7 on her next turn. This will always be possible as 7 is at least half of the number of sweets remaining. Next, player B must leave between 4 and 6 sweets. Player A can then leave 3 sweets as 3 is at least half of the number of sweets remaining. Player B must now take 1 sweet, leaving 2 on the table. Finally, player A takes 1 sweet leaving player B with no valid move.

B5 Find a fraction $\frac{m}{n}$, with m not equal to n, such that all of the fractions

$$\frac{m}{n}, \frac{m+1}{n+1}, \frac{m+2}{n+2}, \frac{m+3}{n+3}, \frac{m+4}{n+4}, \frac{m+5}{n+5}$$

can be simplified by cancelling.

Solution

Suppose that $n > m$ and write $n = m + k$, where k is an integer. Then the six fractions are

$$\frac{m}{m+k}, \frac{m+1}{(m+1)+k}, \frac{m+2}{(m+2)+k}, \frac{m+3}{(m+3)+k}, \frac{m+4}{(m+4)+k}, \frac{m+5}{(m+5)+k}.$$

These fractions can all be cancelled provided that k is a multiple of each of the integers

$$m, m+1, m+2, m+3, m+4, m+5.$$

For example, take $m = 2$. Then k must be a common multiple of 2, 3, 4, 5, 6, 7; say $k = 420$. Then the six fractions are $\frac{2}{422}, \frac{3}{423}, \frac{4}{424}, \frac{5}{425}, \frac{6}{426}, \frac{7}{427}$; so $m = 2$ and $n = 422$ is a solution.

B6 The sum of four different prime numbers is a prime number. The sum of some pair of the numbers is a prime number, as is the sum of some triple of the numbers. What is the smallest possible sum of the four prime numbers?

Solution

One of the four primes must be 2. This is because the sum of four odd positive integers is even and bigger than 2, so cannot be prime. Similarly, 2 must be used in the pair. But 2 must not be used in the triple, otherwise its sum would be even and greater than 2.

The triple must sum to a prime that is also 2 smaller than a prime, so that the four chosen numbers sum to a prime. The sum of the three smallest odd primes is $3 + 5 + 7 = 15$, which is not prime, and so the sum of the triple must be greater than 15. The possible sums are therefore 17, 29, 41, ... In order to have sum 17, one of the numbers 3, 5 or 7 must be increased by 2. However, 3 and 5 cannot be increased by 2 as this would mean the primes in the triple are not all different, and 7 cannot be increased by 2 as 9 is not prime. Thus the triple cannot have sum 17. It is possible, however, to find three primes that sum to 29. For example, 5, 7 and 17.

Therefore the smallest possible sum of the four primes is $29 + 2 = 31$. (And an example of four primes with all of the desired properties is $\{2, 5, 7, 17\}$; the pair could then be $\{2, 5\}$and the triple $\{5, 7, 17\}$.)