SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 28th November 2014

Organised by the United Kingdom Mathematics Trust

The Senior Kangaroo paper allows students in the UK to test themselves on questions set for the best school-aged mathematicians from across Europe and beyond.

RULES AND GUIDELINES (to be read before starting):

1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. **Use B or HB pencil only** to complete your personal details and record your answers on the machine-readable Answer Sheet provided. **All answers are written using three digits, from 000 to 999.** For example, if you think the answer to a question is 42, write 042 at the top of the answer grid and then code your answer by putting solid black pencil lines through the 0, the 4 and the 2 beneath.

   Please note that the machine that reads your Answer Sheet will only see the solid black lines through the numbers beneath, not the written digits above. You must ensure that you code your answers or you will not receive any marks. There are further instructions and examples on the Answer Sheet.
5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

Enquiries about the Senior Kangaroo should be sent to:

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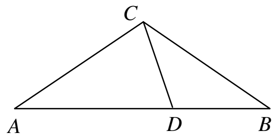
1. Three standard dice are stacked in a tower so that the numbers on each pair of touching faces add to 5. The number on the top of the tower is even. What is the number on the base of the tower?

2. How many prime numbers p have the property that $p^4 + 1$ is also prime?

3. Neil has a combination lock. He knows that the combination is a four-digit number with first digit 2 and fourth digit 8 and that the number is divisible by 9. How many different numbers with that property are there?

4. In the diagram, triangle ABC is isosceles with $CA = CB$ and point D lies on AB with $AD = AC$ and $DB = DC$. What is the size in degrees of angle BCA?

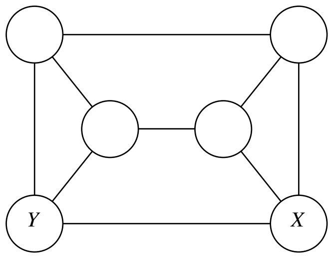


5. Six of the seven numbers 11, 20, 15, 25, 16, 19 and 17 are divided into three groups of two numbers so that the sum of the two numbers in each group is the same. Which number is not used?

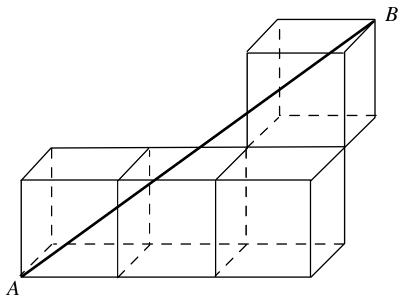
6. The numbers x, y and z satisfy the equations $x^2yz^3 = 7^3$ and $xy^2 = 7^9$. What is the value of $\frac{xyz}{7}$?

7. A table of numbers has 21 columns labelled 1, 2, 3, ..., 21 and 33 rows labelled 1, 2, 3, ..., 33. Every element of the table is equal to 2. All the rows whose label is not a multiple of 3 are erased. All the columns whose label is not an even number are erased. What is the sum of the numbers that remain in the table?

8. Andrew wishes to place a number in each circle in the diagram. The sum of the numbers in the circles of any closed loop of length three must be 30. The sum of the numbers in the circles of any closed loop of length four must be 40. He places the number 9 in the circle marked X. What number should he put in the circle marked Y?



9. Each of the cubes in the diagram has side length 3 cm. The length of *AB* is $\sqrt{k}$ cm. What is the value of *k*?

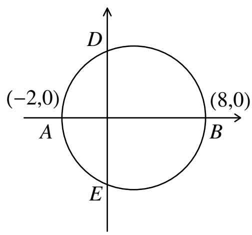


10. A Mathematical Challenge consists of five problems, each of which is worth a different whole number of marks. Carl solved all five problems correctly. He scored 10 marks for the two problems with the lowest numbers of marks and 18 marks for the two problems with the highest numbers of marks. How many marks did he score for all five problems?

11. The mean weight of five children is 45 kg. The mean weight of the lightest three children is 42 kg and the mean weight of the heaviest three children is 49 kg. What is the median weight of the children in kg?

12. On Old MacDonald's farm, the numbers of horses and cows are in the ratio 6:5, the numbers of pigs and sheep are in the ratio 4:3 and the numbers of cows and pigs are in the ratio 2:1. What is the smallest number of animals that can be on the farm?

13. The diagram shows a circle with diameter *AB*. The coordinates of *A* are $(-2, 0)$ and the coordinates of *B* are $(8, 0)$. The circle cuts the *y*-axis at points *D* and *E*. What is the length of *DE*?

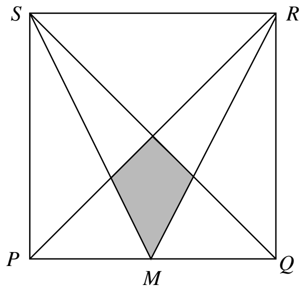


14. Rachel draws 36 kangaroos using three different colours. 25 of the kangaroos are drawn using some grey, 28 are drawn using some pink and 20 are drawn using some brown. Five of the kangaroos are drawn using all three colours. How many kangaroos did she draw that use only one colour?

15. A box contains seven cards numbered from 301 to 307. Graham picks three cards from the box and then Zoe picks two cards from the remainder. Graham looks at his cards and then says "I know that the sum of the numbers on your cards is even". What is the sum of the numbers on Graham's cards?

16. The numbers x, y and z satisfy the equations $x + y + z = 15$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$. What is the value of $x^2 + y^2 + z^2$?

17. In the diagram, *PQRS* is a square. *M* is the midpoint of *PQ*. The area of the square is *k* times the area of the shaded region. What is the value of *k*?



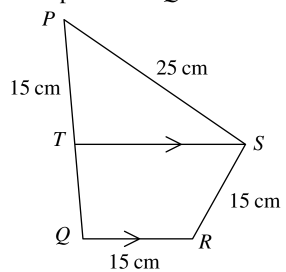
18. Twenty-five workmen have completed a fifth of a project in eight days. Their foreman then decides that the project must be completed in the next 20 days. What is the smallest number of additional workmen required to complete the project on time?

19. In the long multiplication sum shown, each asterisk stands for one digit.

$$\begin{array}{r} *** \\ \times \quad *** \\ \hline 22** \\ 90*0 \\ **2** \\ \hline 56*** \end{array}$$

What is the sum of the digits of the answer?

20. In the quadrilateral *PQRS* with $PQ = PS = 25$ cm and $QR = RS = 15$ cm, point *T* lies on *PQ* so that $PT = 15$ cm and so that *TS* is parallel to *QR*. What is the length in centimetres of *TS*?



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SOLUTIONS

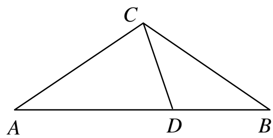
2014 Senior Kangaroo Solutions

1. **5** First note that, on a standard die, the numbers on opposite faces add to 7. Let the number on the top of the tower be n. The numbers on the touching faces going down the tower are then $7 - n$, $5 - (7 - n) = n - 2$, $7 - (n - 2) = 9 - n$ and $5 - (9 - n) = n - 4$ respectively. The bottom number is $7 - (n - 4) = 11 - n$. The numbers on a die are 1 to 6 so $11 - n \leqslant 6$ and hence $n \geqslant 5$. The question states that n is even so $n = 6$. Hence the number on the bottom of the tower is $11 - n = 5$. (It is easy to check that when $n = 6$ all the numbers going down the tower are values that can appear on the face of a standard die.)

2. **1** All prime numbers p greater than 2 are odd. For these numbers, $p^4 + 1$ is even and greater than 2 and so not prime. However, $2^4 + 1 = 17$ which is prime. Hence only one prime number, namely 2, has the desired property.

3. **11** A number is divisible by 9 if and only if the sum of its digits is divisible by 9. Let the second and third digits of the combination be x and y respectively. Hence $10 + x + y$ is divisible by 9. Since $0 \leqslant x \leqslant 9$ and $0 \leqslant y \leqslant 9$ we have $10 + x + y = 18$ or 27. This gives either $x + y = 8$, which has nine different solutions given by $x = 0$, $x = 1$, and so on up to $x = 8$ or $x + y = 17$ which has two different solutions, namely $x = 8$, $y = 9$ and $x = 9$, $y = 8$. This means there are $9 + 2 = 11$ different combinations with the desired property.

4. **108** Let $\angle CAB = x°$. Triangle ABC is isosceles with $CA = CB$ so $\angle CBA = x°$. Triangle BCD is also isosceles with $DB = DC$ so $\angle BCD = x°$.



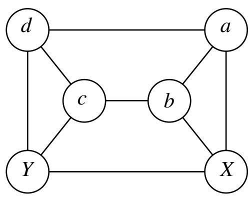
The exterior angle of any triangle is equal to the sum of the interior opposite angles, so $\angle CDA = 2x°$ and hence, since triangle CAD is isosceles, $\angle ACD = 2x°$.
The angle sum of a triangle is 180°, and applying this to triangle CAD we have $x + 2x + 2x = 180$. Therefore $x = 36$ and hence $\angle BCA = 36° + 2 \times 36° = 108°$.

5. **15** Let the number not used be x. The sum of the seven numbers is 123 which is divisible by 3. The six numbers used are divided into three pairs with the same sum so $123 - x$ is also divisible by 3. This means that x is divisible by 3 and the only number in the list that is divisible by 3 is 15. The remaining six numbers can then be paired as 11 and 25, 20 and 16, 19 and 17 all with sum 36.

6. **343** Multiply the two given equations together to obtain $x^2yz^3 \times xy^2 = 7^3 \times 7^9$. Hence $x^3y^3z^3 = 7^{12}$ and so $xyz = 7^4$. Therefore the value of $\frac{xyz}{7}$ is $7^3 = 343$.

7. **220** When the unwanted rows and columns are erased, 11 rows and 10 columns remain. The table then contains 11×10 entries, all equal to 2. Hence the sum of the numbers remaining in the table is $11 \times 10 \times 2 = 220$.

8. **11** Let the numbers placed in the empty circles be a, b, c and d as shown and let y be the number placed in the circle marked Y. Recall that the number placed in the circle marked X is 9. The sum of the numbers in a closed loop of length 3 is 30 so $a + b + 9 = 30$ and $c + d + y = 30$. Add these two equations to get $a + b + 9 + c + d + y = 60$. However, the sum of the numbers in a closed loop of length 4 is 40. Thus we also have $a + b + c + d = 40$. This tells us that $9 + y = 20$ and hence that $y = 11$ so Andrew should place number 11 in the circle marked Y.



9. **153** Use the three-dimensional version of Pythagoras' Theorem to get $AB^2 = (3 \times 3)^2 + (2 \times 3)^2 + (2 \times 3)^2 = 81 + 36 + 36$. Hence $AB^2 = 153$ so $k = 153$.

10. **35** Let the number of marks scored for each question be a, b, c, d and e with $a < b < c < d < e$. The number of marks scored for the two questions with the lowest number of marks is 10 and so $a + b = 10$. However, $a < b$ and so $b \geqslant 6$. Similarly $d + e = 18$ and $d < e$ and hence $d \leqslant 8$. So $6 \leqslant b < c < d \leqslant 8$ and therefore $b = 6$, $c = 7$ and $d = 8$. So the total number of marks Carl scored is $10 + 7 + 18 = 35$.

11. **48** The total weight of the five children is 5×45 kg = 225 kg. Similarly, the total weight of the three lightest children is 3×42 kg = 126 kg and the total weight of the three heaviest children is 3×49 kg = 147 kg. Since there are five children, the child with the median weight is both the third lightest and the third heaviest and so has been included in both of these groups. Hence the median weight is $126\text{kg} + 147\text{kg} - 225\text{kg} = 48\text{kg}$.

12. **123** Since the ratio of the numbers of horses to cows is 6 : 5, the number of cows must be a multiple of 5. Since the ratio of the numbers of cows to pigs is 2 : 1, the number of pigs must also be a multiple of 5. Also, since the ratio of the numbers of pigs to sheep is 4 : 3, the number of pigs is a multiple of 4. Hence the number of pigs is a multiple of 20. The smallest multiple of 20 is 20 itself and one can check that 20 pigs is feasible, with the numbers of horses, cows and sheep being 48, 40 and 15 respectively. This gives the smallest number of animals on the farm as 123.

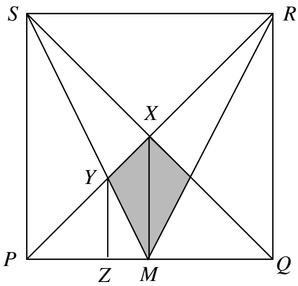
13. **8** The diameter of the circle is $8 - (-2) = 10$ units so the radius is 5 units. The centre of the circle is at X, the midpoint of AB, with coordinates (3, 0). Consider triangle OXD where O is the origin. This is a right-angled triangle with one side 3 units and hypotenuse 5 units so has third side 4 units. Thus the coordinates of D are (0, 4) and the coordinates of E will be (0, −4) by symmetry. Hence the length of DE is $4 - (-4) = 8$ units.

14. **4** Each kangaroo is drawn using one, two or three colours. So, for example, the number 25 of kangaroos drawn using some grey includes the kangaroos that are only grey, those that are grey and exactly one other colour and those that are all three colours. Therefore, by adding 25, 28 and 20 we count those kangaroos with just one colour once, we count those that have exactly two colours twice and those that have all three colours three times. Hence $25 + 28 + 20 = 36$ + (number with exactly two colours) + 2 × (number with three colours) which simplifies to $73 = 36$ + (number with exactly two colours) + 2×5. Hence the number drawn with exactly two colours is 27 and so the number drawn with only one colour is $36 - 27 - 5 = 4$.

15. **912** To be certain that the sum of the numbers on Zoe's two cards is even, the four cards that she chose from cannot contain cards of different parity (that is, they are all odd or all even). The original set of seven cards contained four odd-numbered cards and three even-numbered cards, so the only way a set of four cards all with the same parity can remain is if Graham chose the three even-numbered cards. Hence the sum of the numbers on Graham's cards is $302 + 304 + 306 = 912$.

16. 225 Multiply each term of the second equation by xyz to obtain $yz + xz + xy = 0$. Square each side of the first equation to obtain $(x + y + z)^2 = 15^2$. So $x^2 + 2xy + 2xz + y^2 + 2yz + z^2 = 225$ and hence $x^2 + y^2 + z^2 = 225$.

17. 12 Let the length of the sides of the square be 2 units so its area is 4 units². Introduce points X, Y and Z as shown on the diagram where XM and YZ are parallel to SP and let the length of PZ be x units. The triangles PZY, PMX and PQR are all similar and isosceles so $YZ = x$ and $XM = 1$. Also triangles SPM and YZM are similar so $\frac{2}{1} = \frac{x}{1 - x}$ which has solution $x = \frac{2}{3}$. The shaded area is then $2 \times \frac{1}{2} \times 1 \times \left(1 - \frac{2}{3}\right) = \frac{1}{3}$. Hence the area of the square is $4 \div \frac{1}{3} = 12$ times the shaded area and so $k = 12$.



18. 15 The total number of 'man-days' of work required for the project is $5 \times 25 \times 8 = 1000$. The number of 'man-days' completed is $25 \times 8 = 200$ leaving 800 to be completed. To finish this in 20 days requires $800 \div 20 = 40$ workmen and so an extra $40 - 25 = 15$ workmen are required.

19. 16

```
     ***
  ×  ***
    22**
    90*0
   **2**
   56***
```

The figures 2, 0 and 2 in the hundreds column lines 3, 4 and 5 of the calculation are not large enough to create any carry into the thousands column. Hence the first two missing figures in the third row of working must add with 2 and 9 to give 56 and so are 4 and 5. Note also that the final two digits in that row must be zeros from the structure of the sum so the third row of working is 45200. This means that one of the original 3-digit multiplicands is a 3-digit factor of 452 and so is 452, 226 or 113. The first row of the working is a 4-digit number starting 22 and so is 2260 as it is also a multiple of the same multiplicand. This means that this multiplicand is 452 and that the completed sum is as shown on the right.

Hence the sum of the digits of the answer is $5 + 6 + 5 + 0 + 0 = 16$.

```
     452
  ×  125
    2260
    9040
   45200
   56500
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20. 24 Draw in line PR as shown and let X be the point where PR intersects TS. The corresponding sides of ΔPQR and ΔPSR are equal and so ΔPQR and ΔPSR are congruent (SSS). Hence $\angle PRQ = \angle PRS$. Note also that, because TS and QR are parallel, $\angle PRQ = \angle RXS$ since they are alternate angles. This means that $\angle PRS = \angle RXS$ and so ΔXRS is isosceles and hence $XS = RS = 15$ cm.

TX is parallel to QR and so $\angle PTX = \angle PQR$ and $\angle PXT = \angle PRQ$ using corresponding angles. This means that ΔPTX and ΔPQR are similar and so $TX : QR = PT : PQ$ which gives $TX : 15 = 15 : 25$ so $TX = 9$ cm.

Hence $TS = TX + XS = 9\text{ cm} + 15\text{ cm} = 24$ cm.