# STEP Support Programme

## Hints and Partial Solutions for Assignment 9

## Warm-up

**1** **(i)** When showing that two triangles are congruent, you must fully justify your statements. For example, to justify $AB = BC$, you might write:

$AB = BC$ (given).

In the case of triangles $BMA$ and $BMC$, you might write:

$BM$ is common to both triangles.

Don't assume things that are not given, especially not if it is the thing you are supposed to be proving! In this case, do not assume that the triangle has two equal angles: the definition of an isosceles triangle is that is has two equal *sides* (and you are supposed to **prove** that it has two equal angles). You can use "SSS" to show that triangles $BAM$ and $BCM$ are congruent and from this deduce the required results.

**(ii)** The height of the triangle is given by $h = a\sin C$ (or $c\sin A$). Equating the two expressions for the area of the triangle and rearranging gives the first two-thirds of the sine rule.
Having proved that $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$, you don't have to start all over again to show that $\dfrac{\sin C}{c} = \dfrac{\sin B}{b}$; you can state that this second result follows by relabelling (i.e. swapping the letters around) or by symmetry.

**(iii)** Considering triangle $BMA$ (which is right-angled by part **(i)**) we have $BM = \cos\alpha$ and $AM = \sin\alpha$.
Using the sine rule we have $\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$ which gives $\dfrac{\sin 2\alpha}{AC} = \dfrac{\sin(90^\circ - \alpha)}{1}$, and we can also use $\sin(90^\circ - \alpha) = \cos\alpha$.

# Preparation

**2** **(i)** As a general rule, **do not** divide by something that might be zero. This is quite a simple example, but if you divide by $x$ rather than factorising you will loose one of the solutions. Answer: $x = 0$ or $x = -\frac{3}{5}$.

**(ii)** Remember to show key points on your sketch, but do not work out lots of values and plot the graph. The key points here would be the turning points — at $(-2, 17)$ and $(2, -15)$ — and the $y$-intercept at $(0, 1)$. Since the coefficient of $x^3$ is positive we have $y \to +\infty$ as $x \to +\infty$. You should be able to see from your graph that the equation has three real roots.

**(iii)** Note that these cubics (which have no $x$ term) all have a turning point on the $y$-axis. Use Desmos to check your graphs — as long as the turning points are in the correct quadrant and the shape looks like a cubic then that would (probably) be good enough.

**(iv)** The key point here is that cubics have one turning point on either side of the $x$-axis if (and only if) they have three intercepts with the $x$ axis (i.e. the equations have three real roots). Note that cubics must always have at least one real root.

# The STEP question

**3** The turning points are at $(-\frac{2}{3}A, \frac{4}{27}A^3 + B)$ and $(0, B)$.

When $A > 0$ and $B > 0$ both the turning points are above the $x$ axis, with the minimum at $(0, B)$. There is only one real root (one intersection with the $x$ axis).

When $A < 0$ and $B > 0$ the maximum will be at $(0, B)$ (which is above the $x$ axis). The minimum will be below the $x$ axis when $\frac{4}{27}A^3 + B < 0$, giving three real roots, and above the $x$ axis when $\frac{4}{27}A^3 + B > 0$ when there is only one real root. You should sketch two graphs for $A < 0, B > 0$, one for $\frac{4}{27}A^3 + B < 0$ and one for $\frac{4}{27}A^3 + B > 0$.

For the second part, the letters are now $a$ and $b$, not $A$ and $B$, because in the first part there were conditions on the values of $A$ and $B$.

You need to be careful with the direction of the implication here. The statement

"has three distinct real roots **if** $27b^2 + 4a^3b < 0$"
means
"**if** $27b^2 + 4a^3b < 0$ **then** the equation has three distinct real roots".

It is tempting to show that the converse statement is true instead, i.e.

"**if** the equation has three distinct real roots **then** $27b^2 + 4a^3b < 0$"

which is **not** what the question is asking for.

Start with the "if" statement. Factorise the inequality, so that you have:

$$\text{if} \quad 27b(b + \tfrac{4}{27}a^3) < 0$$

and from that you can consider the two cases that occur, (one case is $b > 0$; $(b + \frac{4}{27}a^3) < 0$). You can then draw a sketch for each and show that they do have three distinct real roots.

You then need to repeat for the other inequality $(27b^2 + 4a^3b)$; four graphs are needed to cover all the possibilities (compare with $B > 0$, $A < 0$ in the first part of the question).

You can start by considering the turning points, assuming that there are three roots. This is a dangerous approach (as you could end up only proving the converse statement), but can be made to work with careful use of "if and only if".

To do this, you might start by showing that:

the equation has three distinct real roots
**if and only if**
the turning points are on opposite sides of the $y$-axis

You will need to sketch graphs of cubics showing all the different cases to justify this. Once you have done this you can say:

the turning points are on opposite sides of the $y$-axis
**if and only if**
the $y$ coordinates $y_1, y_2$ of the turning points have different signs
**if and only if**
$y_1 \times y_2 < 0$

and then use your $y_1$ and $y_2$ to complete the argument.

## Warm down

**4** Euclid did not have the SSS condition for congruence at this point in his book, so for this question we restricted ourselves to SAS.

If we do allow ourselves the SSS condition, then the easiest way to prove that an isosceles triangle has two angles is as in question **1(i)**.

Start by showing that $\triangle BCD$ is congruent to $\triangle BAE$ by using SAS You are given that $BA = BC$, and since the extensions $AD$ and $CE$ are the same length we also have $BE = BD$. The angle at $B$ is a shared angle, so $\angle ABE = \angle CBD$.

Having shown $\triangle BCD$ is congruent to $\triangle BAE$ you now know that $\angle BCD = \angle BAE$, $DC = AE$ and $\angle ADC = \angle AEC$.

Now use SAS again with $\triangle ADC$ and $\triangle CEA$.

It may find it helpful to name the angles (for example: “Let $\angle BAE = \angle BCD = \beta$”) and mark them on your diagram.