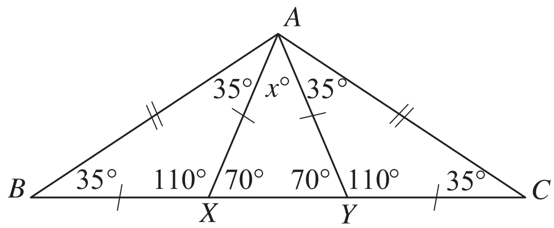
UK Junior Mathematical Olympiad 2016 Solutions

A1 00 Since the integers are consecutive and one ends in a 5, the other integer ends in a 4 or a 6 and so is even. Hence the product of the two numbers is a multiple of 10. When this is squared, we obtain a multiple of 100 and so the final digits are 00.

A2 40



Using the angle sum of a triangle, the isosceles triangles ABX and ACY have angles of 35°, 35° and $180° - 35° - 35° = 110°$. Angles on a straight line add to 180° and so $\angle AXY = \angle XYA = 180° - 110° = 70°$. Using the angle sum of a triangle again, we have $x = 180 - 70 - 70 = 40$.

Alternative Solution

Since $\triangle ABC$ is isosceles, $\angle ACB = 35°$.

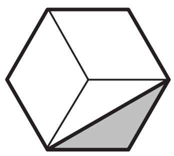
Since $\triangle ABX$ is isosceles, $\angle BAX = 35°$.

Since $\triangle ACY$ is isosceles, $\angle CAY = 35°$.

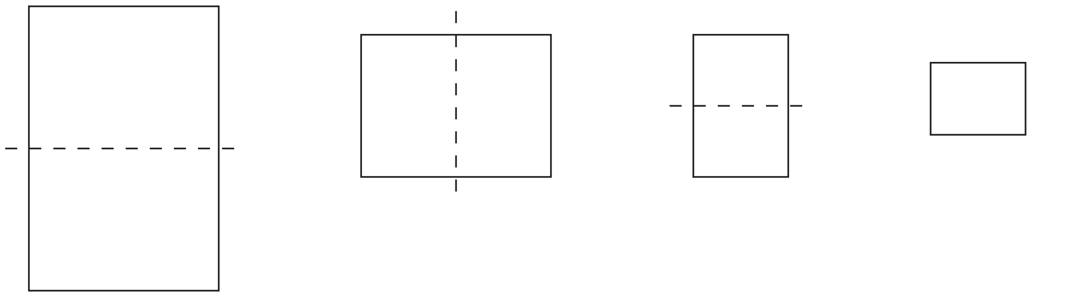
Hence, from the angle sum in $\triangle ABC$ we have $x = 180 - 4 \times 35 = 40$.

A3 123456789 The sequence is 0, 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111, … . Hence the sum of the first 10 terms is 123456789.

A4. 8 m^2 From the dissection shown, the area of the triangle is half of the area of a parallelogram which is itself a third of the area of the hexagon. So the triangle has area $\frac{1}{2} \times \frac{1}{3} \times 48 \text{ m}^2 = 8 \text{ m}^2$.



A5. 35 cm Each fold reduces the area by half. Therefore we require three folds to reduce the area from 600 cm^2 to 75 cm^2.



30 cm × 20 cm 15 cm × 20 cm 15 cm × 10 cm 7.5 cm × 10 cm

From the diagrams, we see that we have, in turn, areas of 600 cm^2, 300 cm^2, 150 cm^2 and 75 cm^2. The final shape has perimeter $2(7.5 + 10) \text{ cm} = 35 \text{ cm}$.

A6. 1:32 We have $AB : BC = 1 : 2$

and $BC : CD = 1 : 3 = 2 : 6$

and $CD : DE = 1 : 4 = 6 : 24$

and so $AB : BC : CD : DE = 1 : 2 : 6 : 24.$

Hence we have $AB : BE = 1 : (2 + 6 + 24) = 1 : 32$

A7. 192 We have $15 = 1 \times 3 \times 5$ and $21 = 1 \times 3 \times 7$. The integer we require is therefore a multiple of $1 \times 3 \times 5 \times 7 = 105$. However, 105 has the eight factors 1, 3, 5, 7, 15, 21, 35 and 105 and any (larger) multiple of 105 would have more than eight factors. Hence 105 is the integer we require and the sum of its factors is

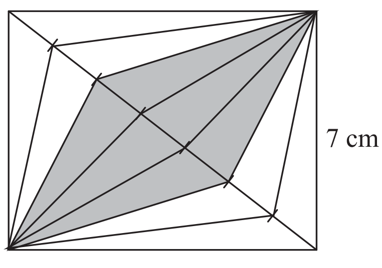
$$1 + 3 + 5 + 7 + 15 + 21 + 35 + 105 = 192.$$

A8. 59 Suppose this happens in x years' time. Julie's age will be $32 + x$, Megan's age will be $4 + x$ and Zoey's age will be $1 + x$. Thus $1 + x + 4 + x = 32 + x$ and, subtracting $5 + x$ from each side of this equation, we have $x = 27$.

In 27 years' time, Julie will be 59, Megan 31 and Zoey 28. Notice that $31 + 28 = 59$ as required.

A9. 30π Each region has a perimeter consisting of the perimeters of a third of a circle of radius 18 cm and two semicircles of radius 9 cm. Using the fact that the perimeter of a circle is π times its diameter, the required perimeter (in cm) is $\frac{1}{3} \times \pi \times 36 + 2 \times \frac{1}{2} \times \pi \times 18 = 30\pi$.

A10. 27 The 14 small triangles shown have equal heights and bases of equal length. Thus they have equal areas and their total area is that of the rectangle. The area of the rectangle is $7\,\text{cm} \times 9\,\text{cm} = 63\,\text{cm}^2$.



The area we require is made up of six small triangles.
Hence, the required area is $\frac{6}{14} \times 63\,\text{cm}^2 = 27\,\text{cm}^2$.

B1 In a certain triangle, the size of each of the angles is a whole number of degrees. Also, one angle is 30° larger than the average of the other two angles.

What is the largest possible size of an angle in this triangle?

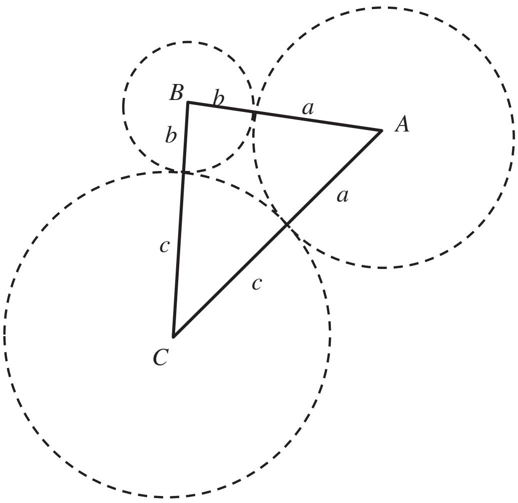
Solution

Let two of the angles measure $a°$ and $b°$. Then the third angle is $\left(30 + \frac{1}{2}(a + b)\right)°$. The angle sum of a triangle is 180° which gives $180° = \left(a + b + 30 + \frac{1}{2}(a + b)\right)° = \left(30 + \frac{3}{2}(a + b)\right)°$ and so $\frac{3}{2}(a + b) = 150$, giving $a + b = 100$. The sizes of all the angles are integers so that the largest either a or b can be is 99. This gives a triangle with angles 1°, 80° and 99° and so the largest possible such angle is 99°.

B2 The points A, B and C are the centres of three circles. Each circle touches the other two circles, and each centre lies outside the other two circles. The sides of the triangle ABC have lengths 13 cm, 16 cm and 20 cm.

What are the radii of the three circles?

Solution



Since the circles touch, for each pair of circles, the distance between their centres is the sum of their radii. Let the radii of the three circles (in cm) be a, b and c. Then we can form equations for the lengths of the sides of the triangle:

$$a + b = 13$$
$$b + c = 16$$
$$a + c = 20.$$

Adding these equations together, we obtain

$$2a + 2b + 2c = 49.$$

So $a + b + c = 24\frac{1}{2}$. But $a + b = 13$ giving $c = 11\frac{1}{2}$. Then $b = 16 - 11\frac{1}{2} = 4\frac{1}{2}$ and $a = 20 - 11\frac{1}{2} = 8\frac{1}{2}$.

B3. A large cube is made up of a number of identical small cubes. The number of small cubes that touch four other small cubes face-to-face is 168.

How many small cubes make up the large cube?

Solution

On a face of the large cube, the cubes not on an edge touch five other faces and the internal cubes not on a face touch six other faces. Small cubes which touch four other faces lie along an edge and are not at a corner (those touch 3 other faces). A cube has 12 edges. Now $\frac{168}{12} = 14$ so that means each edge of the large cube has 14 small cubes which touch four other cubes and two small corner cubes. Therefore an edge of the large cube has length 16 and so the total number of small cubes is $16 \times 16 \times 16 = 4096$.

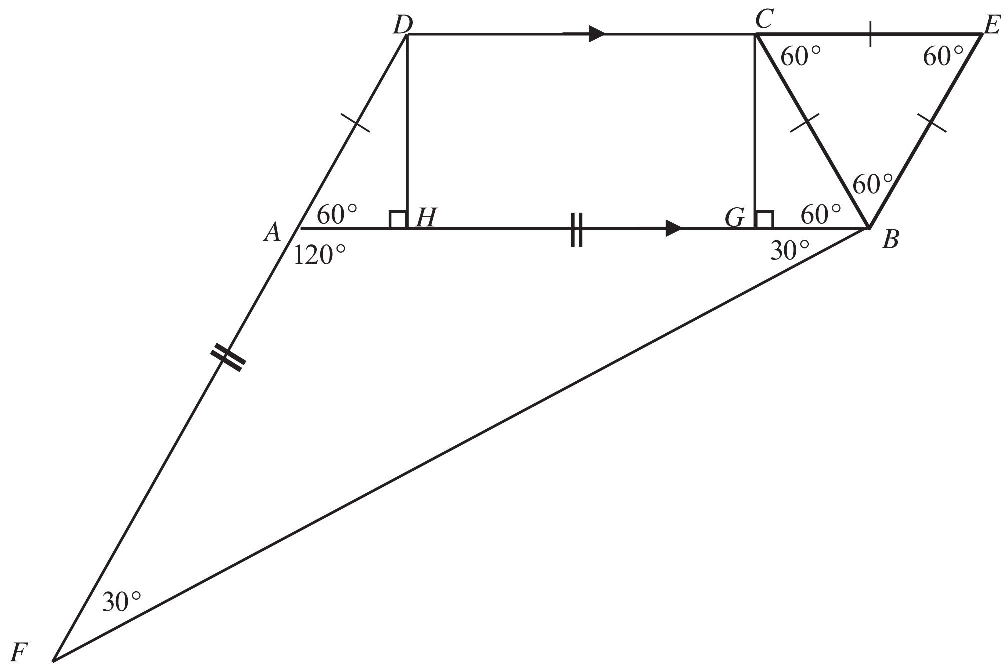
Alternative Solution

The large cube has 8 corner cubes each of which has 3 faces which touch matching cubes. The cubes which lie on the edges between a pair of corner cubes each have 4 faces which touch matching cubes and we know there are 168 of these. A cube has 12 edges so dividing 168 by 12 gives 14 which tells us that each edge of the large cube has 16 small cubes in it. Thus the number of small cubes is $16 \times 16 \times 16 = 4096$.

B4. In the trapezium $ABCD$, the lines AB and CD are parallel. Also $AB = 2DC$ and $DA = CB$. The line DC is extended (beyond C) to the point E so that $EC = CB = BE$. The line DA is extended (beyond A) to the point F so that $AF = BA$.

Prove that $\angle FBC = 90°$.

Solution



Since $EC = CB = BE$, the triangle ECB is equilateral and each of its angles is 60°.

ED and AB are parallel so $\angle ABC = \angle BCE = 60°$ (alternate angles).

Draw perpendiculars from C and D to AB to meet AB at G and H. The right-angled triangles DAH and CBG have hypotenuses of equal length and $DH = CG$ so triangles DAH and CGB are congruent {RHS}. Thus $\angle DAH = \angle CBG = 60°$.

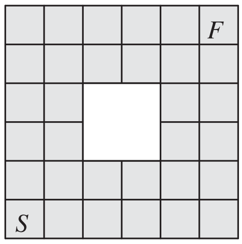
DAF is a straight line so that $\angle BAF = 180° - 60° = 120°$.

Triangle AFB is isosceles with $\angle AFB = \angle FBA = \frac{1}{2}(180° - 120°) = 30°$ (angle sum of a triangle).

Therefore

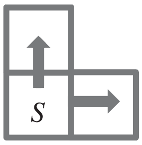
$$\angle FBC = \angle FBA + \angle ABC = 30° + 60° = 90°.$$

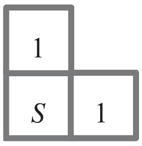
B5. The board shown has 32 cells, one of which is labelled S and another F. The shortest path starting at S and finishing at F involves exactly nine other cells and ten moves, where each move goes from cell to cell 'horizontally' or 'vertically' across an edge.



How many paths of this length are there from S to F?

Solution

From S, there are two starting paths: 

and we can represent the number of paths to reach a square like this: 

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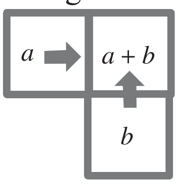
There is only one way to travel along the edges and we can build up our diagram like this, noticing that each square not on an edge can only be reached from the left or from below 

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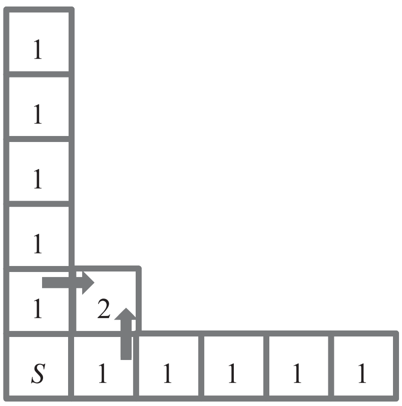

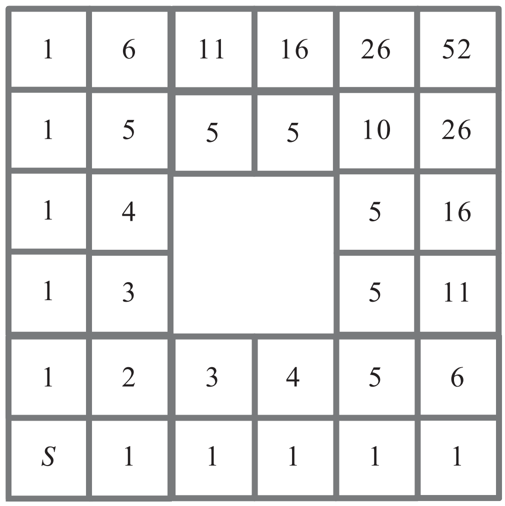

Since F is 5 steps above and 5 steps to the right of S, each of the 10 moves can only be upwards or to the right and so there are 52 paths from S to F.

B6. For which values of the positive integer n is it possible to divide the first $3n$ positive integers into three groups each of which has the same sum?

Solution

$n = 1$: it is impossible to place the integers 1, 2 and 3 into three groups with the same sum.

$n = 2$: $1 + 2 + 3 + 4 + 5 + 6 = 21$ and so we want to place these integers into three groups, each of which add to 7. The groups are (1, 6), (2, 5) and (3, 4).

$n = 3$: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ and so we want to place these integers into three groups, each of which adds to 15. Such a grouping is (1, 2, 3, 4, 5), (6, 9) and (7, 8).

It would not be sensible to continue to look at each value of n, so we look at what happens when we move from $3n$ to $3(n + 2)$.

Let us assume that we can place $3n$ integers into three groups with the same sum. For the first $3(n + 2)$ integers we must include an extra six integers $3n + 1$, $3n + 2$, $3n + 3$, $3n + 4$, $3n + 5$ and $3n + 6$ to deal with $3(n + 2) = 3n + 6$.

When we add the six new integers together, we obtain $18n + 21$ meaning we increase the sum by a third of this, $6n + 7$, to each group to make new groups with the same sum.

So we add pairs to our original three groups: $3n + 1$ with $3n + 6$ to one group; $3n + 2$ with $3n + 5$ to another group and $3n + 3$ with $3n + 4$ to the third group.

However, we know we can obtain three groups with the same sum for $n = 2$ and $n = 3$ and now we can also obtain three groups for each of $n = 4, 6, 8, \ldots$ and so every even integer, in the same way from $n = 3$ we have $n = 5, 7, 9, \ldots$ and so every odd integer greater than 1.

Thus the values of n for which the first $3n$ positive integers can be placed in three groups with the same sum are all values of $n > 1$.