MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON

Wednesday 2 November 2016

Time Allowed: 2½ hours

Please complete the following details in BLOCK CAPITALS. You must use a pen.

Surname	
Other names	

Candidate Number	M					

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial Applicants: if you are applying to Imperial College for any of the Mathematics courses**:** Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted.**

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

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Please complete these details below in block capitals.

Centre Number																
Candidate Number	**M**															
UCAS Number (if known)				–				–								

	d	d		m	m		y	y
Date of Birth			–			–		

Please tick the appropriate box:

- ☐ I have attempted Questions **1,2,3,4,5**
- ☐ I have attempted Questions **1,2,3,5,6**
- ☐ I have attempted Questions **1,2,5,6,7**

Admissions Testing Service

Administered on behalf of the University of Oxford by the Admissions Testing Service, part of Cambridge Assessment, a non-teaching department of the University of Cambridge.

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Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS.**

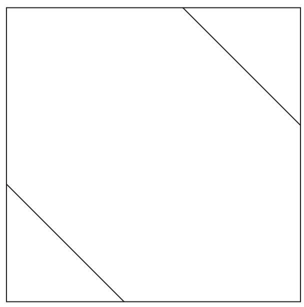
For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

A. A sequence (a_n) has first term $a_1 = 1$, and subsequent terms defined by $a_{n+1} = la_n$ for $n \geqslant 1$. What is the product of the first 15 terms of the sequence?

(a) l^{14}, (b) $15 + l^{14}$, (c) $\dfrac{1 - l^{15}}{1 - l}$, (d) l^{105}, (e) $15 + l^{105}$.

B. An irregular hexagon with all sides of equal length is placed inside a square of side length 1, as shown below (not to scale). What is the length of one of the hexagon sides?



(a) $\sqrt{2} - 1$, (b) $2 - \sqrt{2}$, (c) 1, (d) $\dfrac{\sqrt{2}}{2}$, (e) $2 + \sqrt{2}$.

Turn over

C. The origin lies inside the circle with equation

$$x^2 + ax + y^2 + by = c$$

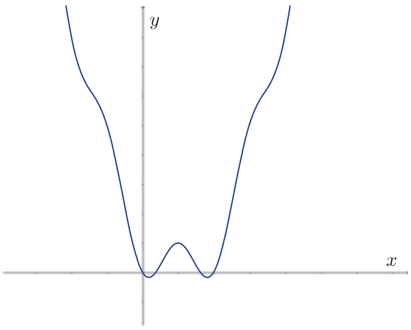
precisely when

(a) $c > 0$, (b) $a^2+b^2 > c$, (c) $a^2+b^2 < c$, (d) $a^2+b^2 > 4c$, (e) $a^2+b^2 < 4c$.

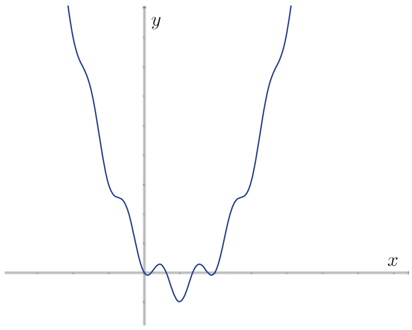
D. How many solutions does $\cos^n(x) + \cos^{2n}(x) = 0$ have in the range $0 \leqslant x \leqslant 2\pi$ for an integer $n \geqslant 1$?

(a) 1 for all n, (b) 2 for all n, (c) 3 for all n,
(d) 2 for even n and 3 for odd n, (e) 3 for even n and 2 for odd n.

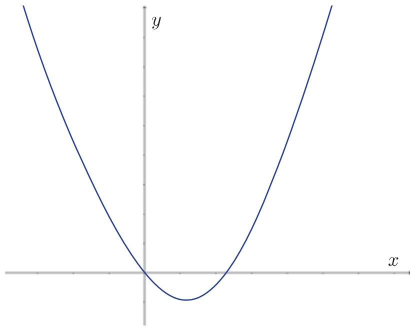
E. The graph of $y = (x-1)^2 - \cos(\pi x)$ is drawn in



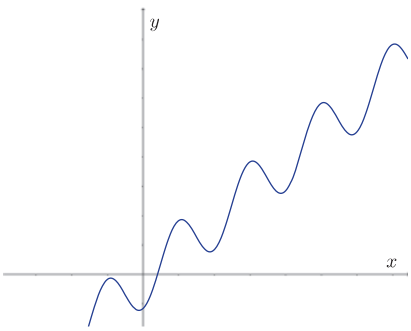
(a)



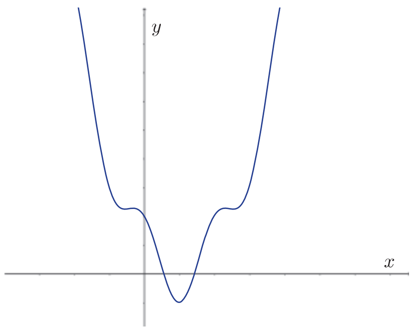
(b)



(c)



(d)



(e)

F. Let n be a positive integer. Then $x^2 + 1$ is a factor of

$$(3 + x^4)^n - (x^2 + 3)^n (x^2 - 1)^n$$

for

(a) all n, (b) even n, (c) odd n, (d) $n \geqslant 3$, (e) no values of n.

Turn over

G. The sequence (x_n), where $n \geqslant 0$, is defined by $x_0 = 1$ and

$$x_n = \sum_{k=0}^{n-1} x_k \qquad \text{for } n \geqslant 1.$$

The sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

equals

(a) 1, (b) $\frac{6}{5}$, (c) $\frac{8}{5}$, (d) 3, (e) $\frac{27}{5}$.

H. Consider two functions

$$f(x) = a - x^2$$
$$g(x) = x^4 - a.$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?

(a) all values of a, (b) $a > 1$, (c) $a > \frac{6}{5}$,

(d) $a > \left(\frac{4}{3}\right)^{\frac{3}{2}}$, (e) $a > \left(\frac{6}{5}\right)^{4}$.

I. Let a and b be positive real numbers. If $x^2 + y^2 \leqslant 1$ then the largest that $ax + by$ can equal is

(a) $\dfrac{1}{a} + \dfrac{1}{b}$, (b) $\max(a, b)$, (c) $\sqrt{a^2 + b^2}$, (d) $a+b$, (e) a^2+ab+b^2.

J. Let $n > 1$ be an integer. Let $\Pi(n)$ denote the number of distinct prime factors of n and let $x(n)$ denote the final digit of n. For example, $\Pi(8) = 1$ and $\Pi(6) = 2$. Which of the following statements is false?

(a) If $\Pi(n) = 1,$ there are some values of $x(n)$ that mean n cannot be prime,
(b) If $\Pi(n) = 1,$ there are some values of $x(n)$ that mean n must be prime,
(c) If $\Pi(n) = 1,$ there are values of $x(n)$ which are impossible,
(d) If $\Pi(n) + x(n) = 2,$ we cannot tell if n is prime,
(e) If $\Pi(n) = 2,$ all values of $x(n)$ are possible.

Turn over

2. For **ALL APPLICANTS.**

Let

$$A(x) = 2x + 1, \qquad B(x) = 3x + 2.$$

(i) Show that $A(B(x)) = B(A(x))$.

(ii) Let n be a positive integer. Determine $A^n(x)$ where

$$A^n(x) = \underbrace{A(A(A \cdots A}_{n \text{ times}}(x) \cdots).$$

Put your answer in the simplest form possible.

A function $F(x) = 108x + c$ (where c is a positive integer) is produced by repeatedly applying the functions $A(x)$ and $B(x)$ in some order.

(iii) In how many different orders can $A(x)$ and $B(x)$ be applied to produce $F(x)$? Justify your answer.

(iv) What are the possible values of c? Justify your answer.

(v) Are there positive integers $m_1, \ldots, m_k, n_1, \ldots, n_k$ such that

$$A^{m_1}B^{n_1}(x) + A^{m_2}B^{n_2}(x) + \cdots + A^{m_k}B^{n_k}(x) = 214x + 92 \qquad \text{for all } x?$$

Justify your answer.

Turn over

If you require additional space please use the pages at the end of the booklet

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

In this question we fix a real number α which will be the same throughout. We say that a function f is **bilateral** if

$$f(x) = f(2\alpha - x)$$

for all x.

(i) Show that if $f(x) = (x - \alpha)^2$ for all x then the function f is bilateral.

(ii) On the other hand show that if $f(x) = x - \alpha$ for all x then the function f is *not* bilateral.

(iii) Show that if n is a non-negative integer and a and b are any real numbers then

$$\int_a^b x^n \, \mathrm{d}x = -\int_b^a x^n \, \mathrm{d}x.$$

(iv) Hence show that if f is a polynomial (and a and b are any reals) then

$$\int_a^b f(x) \, \mathrm{d}x = -\int_b^a f(x) \, \mathrm{d}x.$$

(v) Suppose that f is any bilateral function. By considering the area under the graph of $y = f(x)$ explain why for any $t \geqslant \alpha$ we have

$$\int_\alpha^t f(x) \, \mathrm{d}x = \int_{2\alpha - t}^{\alpha} f(x) \, \mathrm{d}x.$$

If f is a function then we write G for the function defined by

$$G(t) = \int_\alpha^t f(x) \, \mathrm{d}x$$

for all t.

(vi) Suppose now that f is any bilateral polynomial. Show that

$$G(t) = -G(2\alpha - t)$$

for all t.

(vii) Suppose f is a bilateral polynomial such that G is also bilateral. Show that $G(x) = 0$ for all x.

Turn over

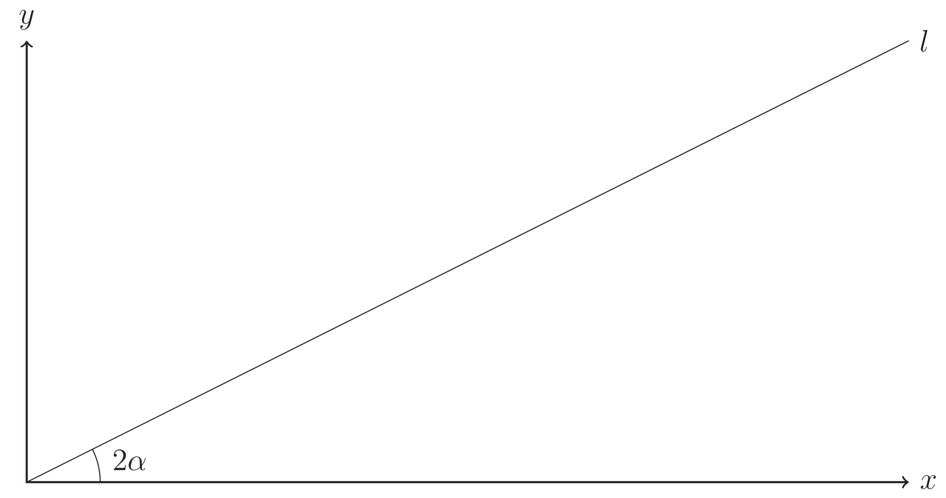
If you require additional space please use the pages at the end of the booklet

4.

For **APPLICANTS IN** { **MATHEMATICS**, **MATHEMATICS & STATISTICS**, **MATHEMATICS & PHILOSOPHY** } **ONLY.**

Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

The line l passes through the origin at angle 2α above the x-axis, where $2\alpha < \frac{\pi}{2}$.



Circles C_1 of radius 1 and C_2 of radius 3 are drawn between l and the x-axis, just touching both lines.

(i) What is the centre of circle C_1?

(ii) What is the equation of circle C_1?

(iii) For what value of α do circles C_1 and C_2 touch?

(iv) For this value of α (for which the circles C_1 and C_2 touch) a third circle, C_3, larger than C_2, is to be drawn between l and the x-axis. C_3 just touches both lines and also touches C_2. What is the radius of this circle C_3?

(v) For the same value of α, what is the area of the region bounded by the x-axis and the circles C_1 and C_2?

Turn over

If you require additional space please use the pages at the end of the booklet

5. For **ALL APPLICANTS.**

This question concerns the sum s_n defined by

$$s_n = 2 + 8 + 24 + \cdots + n2^n.$$

(i) Let $f(n) = (An + B)2^n + C$ for constants A, B and C yet to be determined, and suppose $s_n = f(n)$ for all $n \geqslant 1$. By setting $n = 1, 2, 3$, find three equations that must be satisfied by A, B and C.

(ii) Solve the equations from part (i) to obtain values for A, B and C.

(iii) Using these values, show that if $s_k = f(k)$ for some $k \geqslant 1$ then $s_{k+1} = f(k+1)$.

You may now assume that $f(n) = s_n$ for all $n \geqslant 1$.

(iv) Find simplified expressions for the following sums:

$$t_n = n + 2(n-1) + 4(n-2) + 8(n-3) + \cdots + 2^{n-1}1,$$
$$u_n = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n}.$$

(v) Find the sum

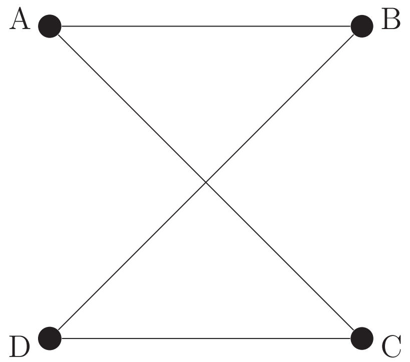
$$\sum_{k=1}^{n} s_k.$$

Turn over

If you require additional space please use the pages at the end of the booklet

6.

For **APPLICANTS IN** { **COMPUTER SCIENCE** / **MATHEMATICS & COMPUTER SCIENCE** / **COMPUTER SCIENCE & PHILOSOPHY** } **ONLY.**



Four people A, B, C, D are performing a dance, holding hands in the arrangement shown above. Each dancer is assigned a 1 or a 0 to determine their steps, and there must always be at least a 1 and a 0 in the group of dancers (dancers cannot all dance the same kind of steps). A dancer is **off-beat** if their assigned number plus the numbers assigned to the people holding hands with them is odd. The entire dance is an **off-beat dance** if every dancer is off-beat.

(i) In how many ways can the four dancers perform an off-beat dance? Explain your answer.

A new dance starts and two more people, E and F, join the dance such that each dancer holds hands with their neighbours to form a ring.

(ii) In how many ways can the ring of six dancers perform an off-beat dance? Explain your answer.

(iii) In a ring of n dancers explain why an off-beat dance can only occur if n is a multiple of 3.

(iv) For a new dance a ring of $n > 4$ dancers, each holds hands with dancers one person away from them round the ring (so C holds hands with A and E and D holds hands with B and F and so on). For which values of n can the dance be off-beat?

On another planet the alien inhabitants have three (extendible) arms and still like to dance according to the rules above.

(v) If four aliens dance, each holding hands with each other, how many ways can they perform an off-beat dance?

(vi) Six aliens standing in a ring perform a new dance where each alien holds hands with their direct neighbours and the alien opposite them in the ring. In how many ways can they perform an off-beat dance?

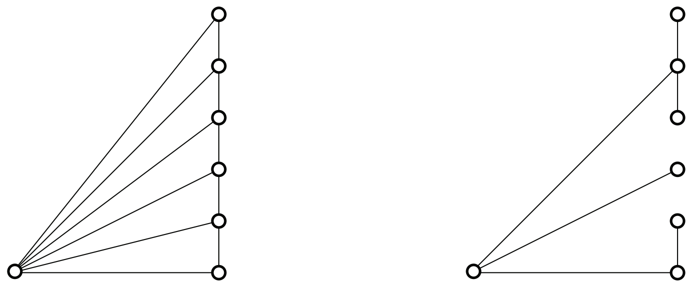
Turn over

If you require additional space please use the pages at the end of the booklet

7.

For **APPLICANTS IN** { **COMPUTER SCIENCE** / **COMPUTER SCIENCE & PHILOSOPHY** } **ONLY.**

An n-**fan** consists of a row of n points, the **tips**, in a straight line, together with another point, the **hub**, that is not on the line. The n tips are joined to each other and to the hub with line segments. For example, the left-hand picture here shows a 6-fan,



For a given n-fan, an n-**span** is a subset containing all $n+1$ points and exactly n of the line segments, chosen so that all the points are connected together, with a unique path between any two points. The right-hand picture shows one of many 6-spans obtained from the given 6-fan; in this 6-span, the tips are in “groups” of 3, 1 and 2, with the top “group” containing 3 tips.

(i) Draw all three 2-spans.

(ii) Draw all 3-spans.

(iii) By considering the possible sizes of the top group of tips and how the group is connected to the hub, calculate the number of 4-spans.

(iv) For $n \geqslant 1$ let z_n denote the number of n-spans. Give an expression for z_n in terms of z_k, where $1 \leqslant k < n$. Use this expression to show that $z_5 = 55$.

(v) Use this relationship to calculate z_6.

End of last question

Turn over

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