SOLUTIONS FOR ADMISSIONS TEST IN

MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS

WEDNESDAY 30 OCTOBER 2019

Mark Scheme:
Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.
Each of Questions 2-7 is worth 15 marks

1

A The cubic on the left-hand side, $x^3 - 300x$, has turning points at $x = \pm 10$, and at $x = -10$ the cubic has value 2000. Since the right-hand side of the equation is greater than 2000, there can only be one root to the equation.
The answer is (b)

B The number 1 is a square, 1 is a cube, and 1×1 is both a cube and a square. 4 is a square, 8 is a cube, but the product 4×8 is neither a square nor a cube.
The answer is (c)

C When $x = 0$, $y = 0+0+0+\cdots = 0$. When $x = 90°$, $y = 1+1+1+\ldots$ diverges. Alternatively, note that this is a geometric series, with sum $\sin^2 x/(1-\sin^2 x) = \tan^2 x$.
The answer is (d)

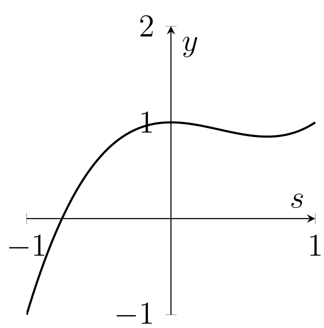
D

$$\left|\int_{-a}^{0} \left|(x^2+2ax+a)-(a-x^2)\right| \,\mathrm{d}x\right| = \left|\left[\left|\frac{2x^3}{3}+ax^2\right|\right]_{-a}^{0}\right| = \left|\frac{a^3}{3}\right|$$

so the integral is 9 for $a = 3$ or $a = -3$.
The answer is (b)

E The graph includes the lines $x = y$, $x = y + 360°$, $x = y + 720°$ and so on.
The answer is (e)

F Use $\cos^2 x + \sin^2 x = 1$. The equation becomes $\sin^3 x + 1 - \sin^2 x = 0$. Write s for $\sin x$. The cubic $s^3 - s^2 + 1$ is -1 at $s = -1$, is 1 at $s = 0$, and is 1 at $s = 1$. The turning points are at $s = 0$ and $s = 2/3$, and the value at $2/3$ is $23/27 > 0$. So this cubic has one (negative) root.

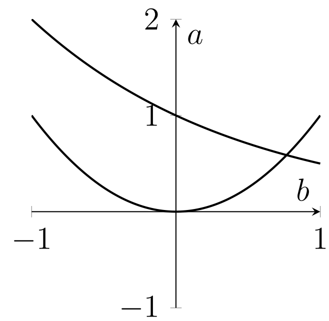


There are two solutions for $\sin x = s$ in the given range.
The answer is (c)

G The equations are equivalent to;

$$b = a^c, \qquad a = b^{c+3/2}, \qquad a = c^b$$

Raise each side of the second equation to the power of c. Then $b = b^{c^2+3c/2}$, so we must have $1 = c^2 + 3c/2$ (since $b > 1$ for the logarithms to make sense). This has solutions $c = -2$ or $c = -1/2$, but we know $c > 0$ so $c = 1/2$. The remaining equations read $a = b^2$ and $a = 2^{-b}$. Sketch both curves;



There is exactly one positive solution.
The answer is (a)

H The hypotenuse is the longest side, so write it as ar^2, where $a > 0$ is the shortest side and $r > 1$. The remaining side must be ar for the sides to be in geometric progression. Then Pythagoras' theorem gives $1 + r^2 = r^4$. Solve the quadratic for r^2 to get $r^2 = (1 + \sqrt{5})/2$, taking the positive root since $r^2 > 0$. Now $r > 0$ so $r = \sqrt{(1+\sqrt{5})/2}$. $\tan \angle BAC$ is either r or $1/r$ depending on which way round A and C are.
The answer is (c)

I Both x and 2^x are strictly increasing for $x > 0$, so their product is strictly increasing. $x2^x$ therefore takes no values twice for $x > 0$, so there are no solutions to the given equation with $0 < x < y$.
The answer is (a)

J Define co-ordinates with the origin is at O, and A on the positive x-axis. Let θ be the angle the line makes with the x-axis. The length of PQ is a function of θ, call it $f(\theta)$. This function is even $f(-\theta) = f(\theta)$ because the triangle has reflectional symmetry in the x-axis. The function therefore has a stationary value at $\theta = 0$. The function is also periodic with period 60°. Combining these facts, we see that $f(\theta)$ must take a stationary value halfway between the stationary points at $\theta = 30°$, i.e. parallel to one of the sides. The length of PQ in this configuration is $2/3$.
The answer is (d)

2.

(i) $N = 1 + 2 + 3 + \cdots + k = k(k+1)/2$. **2 marks**

(ii) Setting $x = 1$ we have

$$2^k = a_0 + a_1 + a_2 + \cdots + a_{N-1} + a_N \leqslant (N+1)a_{\max}.$$

Hence $a_{\max} \geqslant 2^k/(N+1)$ and the result follows. **3 marks**

(iii) Once $k \geqslant i$ then a_i becomes constant. This is because every further bracket, after $(1 + x^i)$ includes powers of x too large to contribute to the coefficient x^i.
3 marks

(iv) Replacing x with x^{-1} we have

$$(1 + x^{-1})(1 + x^{-2}) \times \cdots \times (1 + x^{-k}) = a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots + a_N x^{-N}.$$

Multiplying by x^N we find

$$\begin{aligned} & a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} + a_N x^N \\ = \; & (x^k + 1) \times \cdots \times (x^2 + 1)(x + 1) \\ = \; & a_0 x^N + a_1 x^{N-1} + \cdots + a_{N-1} x + a_N \end{aligned}$$

and comparing coefficients gives $a_i = a_{N-i}$ for all i. **3 marks**

(v) There are $N + 1$ coefficients and the largest coefficient is at least $2^k/(N+1)$. So if all whole numbers up to $a_{\max}$ are to appear then we need that $a_{\max} \leqslant N + 1$. This leads to the inequality

$$2^k \leqslant (N+1)^2 = \left(\frac{k(k+1)}{2} + 1\right)^2.$$

Now exponentials ultimately grow faster than polynomials so this cannot hold for large k. **4 marks**

3.

(i) We have

$$\int_0^c x(c-x)\,\mathrm{d}x = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}.$$

If we choose $c = b - a > 0$ then this integral gives the area of a region that is a translation by a to the left of region S and so has the same area.

3 marks

(ii) The tangency of the line means the quadratic

$$(x-a)(b-x) = mx$$

has a repeated root. This quadratic's discriminant is

$$(a+b-m)^2 - 4ab.$$

Setting this to zero and solving for m gives

$$m = a + b \pm 2\sqrt{ab} = \left(\sqrt{b} \pm \sqrt{a}\right)^2.$$

The greater root of m corresponds to a line meeting the parabola tangentially in the third quadrant, so we want the smaller root.

5 marks

(iii) The areas are equal when

$$\frac{(2\beta+1)(\beta-1)^2}{6} = \frac{(\beta^2-1)^3}{6}.$$

This rearranges to

$$\begin{aligned} 0 &= \frac{(\beta-1)^2}{6}\left[(\beta^2-1)(\beta^2+2\beta+1) - 2\beta - 1\right] \\ &= \frac{(\beta-1)^2}{6}\left[\beta^4 + 2\beta^3 - 4\beta - 2\right] \end{aligned}$$

as required. Note that the quartic factor is negative when $\beta = 1$ but will tend to infinity as $\beta \to \infty$, so there is a root $\beta > 1$, which corresponds to equal areas.

Give this value of β found above the name β^*. Take the graph with equal areas for $a = 1$ and $\beta = \beta^*$, and stretch parallel to the x-axis by a scale factor of a. This will preserve tangency and ratio of areas, while the roots will now be a and $b = \beta^* a > a$. (We should also stretch parallel to the y-axis, to make sure that the leading coefficient of the parabola is still -1, but candidates were not expected to spot this).

7 marks

4.

(i) The nearest points to (a,b) is

$$\begin{array}{ccc} \dfrac{(a,b)}{\sqrt{a^2+b^2}} & \text{when} \quad a^2+b^2 & > & 1. \\ (a,b) & \text{when} \quad a^2+b^2 & \leqslant & 1. \end{array}$$

3 marks

(ii) Consider the open disc $x^2+y^2<1$ and any point (a,b) outside the disc.
Alternative: any other working example.

1 mark

(iii) Consider the circle $x^2+y^2=1$ and $(a,b)=(0,0)$.
Alternative: any other working example.

1 mark

(iv) A general point of S is $(x, mx+c)$ and if D is the distance from (a,b) then

$$D^2 = (x-a)^2 + (mx+c-b)^2.$$

This is a quadratic in x with a positive leading coefficient and so has a unique minimum. (Or this might be shown by differentiating.) **3 marks**

(v) If

$$(x,y) = \left(\frac{a+2b-2}{5}, \frac{2a+4b+1}{5}\right)$$

then

$$a+2b = 5x+2, \qquad 2a+4b = 5y-1$$

and so $2(5x+2) = (5y-1)$ or equally $y = 2x+1$. Moreover, as we vary a from $-\infty$ to ∞, we get every point on this line, so the entire line is in S. **4 marks**

(vi) Suppose that P, Q are two points of S which are both nearest points. The square of the distance of (a,b) from the line segment PQ is a quadratic in x with positive leading coefficient – as seen in (iv). If such a quadratic takes equal values at end points of an interval, then it achieves a strictly lower minimum between these two values on the line segment PQ and such a point would now be a nearer point of S. **3 marks**

5.

(i) To form k subsets as specified, we need at least $2k$ elements. If $k > n/2$ then $2k > n$ and we don't have enough. **2 marks**

(ii) $f(n, 1) = 1$ if $n \geq 2$, and $f(1, 1) = 0$. All the elements are in the same subset, with no choices to make. If $n = 1$, even this is disallowed. **2 marks**

(iii) If we pair $n + 1$ with one of the n elements r of $S = \{1, 2, \ldots, n\}$, then we can continue with any of the $f(n - 1, k - 1)$ partitions of $S \backslash r$ into $k - 1$ subsets. Otherwise, we can add $n + 1$ to one of the k subsets in one of the $f(n, k)$ partitions of S, getting a different partition each time. Thus if $2 \leq k < n$,

$$f(n + 1, k) = nf(n - 1, k - 1) + kf(n, k).$$

4 marks

(iv) We can complete (the starred items in) the table,

n\k	1	2	3	4
1	0			
2	1	0		
3	1	0	0	
4	1	3*	0	0
5	1	10*	0	0
6	1	25	15*	0
7	1	56	105*	0

to find that $f(7, 3) = 105$. In detail:

- $f(4, 2) = 3f(2, 1) = 3$,
- $f(5, 2) = 4f(3, 1) + 2f(4, 2) = 10$ as given,
- $f(6, 3) = 5f(4, 2) = 15$,
- $f(7, 3) = 6f(5, 2) + 3f(6, 3) = 60 + 45 = 105$. **4 marks**

(v) To get a partition of $2n$ items into n pairs, we can arrange them in any of $(2n)!$ orders, then pair each item with its neighbour. This overcounts by a factor of $n!$ because we can shuffle the pairs, and another factor of 2^n because we can swap the elements of each pair. Answer: $(2n)!/n!\, 2^n$.

Alternatively, because $f(2n + 1, n + 1) = 0$, our recurrence reduces to $f(2n + 2, n + 1) = (2n + 1)f(2n, n)$ with $f(2, 1) = 1$, whence

$$f(2n, n) = (2n - 1)(2n - 3) \ldots 3.1,$$

and $f(2n, n) = (2n)!/n!\, 2^n$ as before. (*Accept either* $(2n)!/n!\, 2^n$ *or* $(2n - 1)(2n - 3) \ldots 1$.)
3 marks

6.

(i) $5 = \langle 21\rangle$, $6 = \langle 100\rangle$, $7 = \langle 101\rangle$, $8 = \langle 110\rangle$, $9 = \langle 111\rangle$, $10 = \langle 120\rangle$, $11 = \langle 121\rangle$, $12 = \langle 200\rangle$, $13 = \langle 201\rangle$. **2 marks**

(ii) Note that $\langle 1\rangle = 1$, $\langle 10\rangle = 2$, $\langle 100\rangle = 6$, $\langle 1000\rangle = 24$, and in general digit k counting from the right has place-value $k!$. So multiply each digit by the appropriate factorial and add up the results. Thus,

$$\langle 1221\rangle = 1 \times 24 + 2 \times 6 + 2 \times 2 + 1 \times 1 = 41.$$

Alternatively, if the leftmost digit appears in column k counting from 1 at the right, multiply that digit by k, add the digit next to the right, multiply by $k - 1$ and so on. Thus,

$$\langle 1221\rangle = (1 \times 4 + 2) : \langle 21\rangle = (6 \times 3 + 2) : \langle 1\rangle = 20 \times 2 + 1 = 41.$$

2 marks

(iii) Form a descending sequence of factorials, beginning with the largest that does not exceed the input. Divide the input by each factorial in turn, writing down the quotient as the next digit and keeping the remainder to use as input to the next division. Thus $255 = \langle 20211\rangle$, because

$$\begin{aligned} 255 &= 2 \times 120 + 15 \\ 15 &= 0 \times 24 + 15 \\ 15 &= 2 \times 6 + 3 \\ 3 &= 1 \times 2 + 1 \\ 1 &= 1 \times 1 + 0. \end{aligned}$$

Alternatively, divide the input by 2, and write down the remainder as the rightmost digit. Take the quotient and divide it by 3, writing down the remainder as the next digit, then divide by 4, and so on, until the quotient is zero. Thus

$$\begin{aligned} 255 &= 127 \times 2 + 1 \\ 127 &= 42 \times 3 + 1 \\ 42 &= 10 \times 4 + 2 \\ 10 &= 2 \times 5 + 0 \\ 2 &= 0 \times 6 + 2. \end{aligned}$$

2 marks

(iv) Add the digits from right to left. If the sum of the two digits in column k (plus the carry if any) is more than k, then reduce it by $k + 1$ and carry one to the left. Add leading zeros to each input if the other is longer, or if needed to deal with a final carry. For example,

```
  4 3 2 1
----------
 <1 2 2 1>            <1221> = 24+12+4+1  = 41
   <2 0 1>             <201> = 12+1       = 13
----------
  1 1 1     Carries
----------
 <2 1 0 0>  Result    <2100> = 48+6       = 54
```

4 marks

(v) A flexnum is divisible by 3 if it ends in 00 or 11, because all digits from the third leftwards have a place-value divisible by 6, and 0 and 3 are $\langle 00 \rangle$ and $\langle 11 \rangle$. **2 marks**

(vi) Take the number $\langle 34101 \rangle$ one digit at a time.

- 3: We can divide the 720 permutations into groups of 120 according to their first letter, and the permutation numbered $\langle 34101 \rangle$ will be in the fourth group (numbered 3), so it begins d.
- 4: The remainder is the $\langle 4101 \rangle$'th permutation of $abcef$: it belongs to the fifth group of 24 and starts with f.
- 1: Next, from $abce$ we choose item $1 = b$.
- 0: From ace we choose item $0 = a$.
- 1: From ce we choose item $1 = e$.
- We are left with c.

Answer: *dfbaec*. **3 marks**

7.

(i) The box has finitely many states, so eventually some state must occur for a second time. **1 mark**

(ii) $x_{i+s} = f^s(x_i) = f^s(x_j) = x_{j+s}$. **2 marks**

(iii) We have $x_m = x_{m+p}$, so if $i = m + s$ then $x_i = x_{m+s} = x_{m+p+s} = x_{i+p}$, and (applying this k times) $x_i = x_{i+p} = x_{i+2p} = \cdots = x_{i+kp}$. **2 marks**

(iv) [Possible hint: let $j - i = kp + r$ where $0 \le r < p$ and show that if $x_i = x_j$ then $r = 0$.]

If $i < j$ and the condition holds, say $j - i = kp$, then $x_j = x_{i+kp} = x_i$ by part (iii).

Conversely, if $x_i = x_j$ with $i < j$, then $i \ge m$ by the definition of m.

Now suppose $j - i = kp + r$ with $0 \le r < p$: we will show $r = 0$.

If $x_i = x_j$ then also $x_{i+s} = x_{j+s}$, and we may choose s so that $i + s - m$ is a multiple of p, say $i + s = m + hp$ and $j + s = m + (h + k)p + r$, so that $x_m = x_{i+s}$ and $x_{m+r} = x_{j+s}$.

But then $x_m = x_{i+s} = x_{j+s} = x_{m+r}$. The smallest positive solution of $x_m = x_{m+r}$ is $r = p$, but we know $r < p$, and it follows that $r = 0$ and $j - i$ is a multiple of p. **4 marks**

(v) We must have $u \ge m$ and u a multiple of p. The smallest u for which this happens is the least multiple of p that is at least m, i.e., $u = p\lceil m/p \rceil$. (*Ceiling notation not required.*) **2 marks**

(vi) After resetting, press button B on one box u times, then start pressing button B once on each box and comparing the results. We find $x_i = x_{i+u}$ when $i = m$ and not before.

Alternatively, continue the previous experiment by resetting the box with $2u$ presses, then proceed as above. **2 marks**

(vii) After resetting, press button B on both boxes m times, then start pressing button B on one box and comparing. We find $x_m = x_{m+i}$ when $i = p$ and not before.

Alternatively, continue the previous experiment by pressing button B on one box until the two displays agree again. **2 marks**