



Imperial College
London



MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON and/or UNIVERSITY OF WARWICK

November 2021

Time Allowed: 2½ hours

Please complete the following details in BLOCK CAPITALS. You must use a pen.

Surname					
Other names					
Candidate Number	M				

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5**.
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6**.
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7**.

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial or Warwick Applicants: if you are applying to the University of Warwick for Mathematics BSc, Master of Mathematics, or if you are applying to Imperial College for any of the Mathematics courses: Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year Abroad, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, you should attempt Questions **1,2,3,4,5**.

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted.**

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

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USE ONLY

Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For ALL APPLICANTS.

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

A. A regular dodecagon is a 12-sided polygon with all sides the same length and all internal angles equal. If I construct a regular dodecagon by connecting 12 equally-spaced points on a circle of radius 1, then the area of this polygon is

- (a) $6 + 3\sqrt{3}$, (b) $2\sqrt{2}$, (c) $3\sqrt{2}$, (d) $3\sqrt{3}$, (e) 3.

B. The positive number a satisfies

$$\int_0^a (\sqrt{x} + x^2) \, dx = 5$$

if

- (a) $a = (\sqrt{21} - 1)^{1/3}$, (b) $a = \sqrt{3}$, (c) $a = 3^{2/3}$,
(d) $a = (\sqrt{6} - 1)^{2/3}$, (e) $a = 5^{2/3}$.

Turn over

C. Tangents to $y = e^x$ are drawn at (p, e^p) and (q, e^q) . These tangents cross the x -axis at a and b respectively. It follows that, for all p and q ,

(a) $pa = qb$,

(b) $p - a < q - b$,

(c) $p - a = q - b$,

(d) $p - a > q - b$,

(e) $p + q = a + b$.

D. The area of the region bounded by the curve $y = e^x$, the curve $y = 1 - e^x$, and the y -axis equals

(a) 0, (b) $1 - \ln 2$, (c) $\frac{1}{2} - \frac{1}{2} \ln 2$,
(d) $\ln 2 - 1$, (e) $1 - \ln \frac{1}{2}$.

[Note that $\ln x$ is alternative notation for $\log_e x$.]

E. Six vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ are each chosen to be either $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ with equal probability, with each choice made independently. The probability that the sum $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 + \mathbf{v}_6$ is equal to the vector $\begin{pmatrix} 10 \\ 8 \end{pmatrix}$ is

- (a) 0, (b) $\frac{3}{64}$, (c) $\frac{15}{64}$, (d) $\frac{1}{6}$, (e) $\frac{5}{16}$.

F. The tangent to the curve $y = x^3 - 3x$ at the point $(a, a^3 - 3a)$ also passes through the point $(2, 0)$ for precisely

- (a) no values of a ,
 (b) one value of a ,
 (c) two values of a ,
 (d) three values of a ,
 (e) all values of a .

Turn over

G. The sum

$$\sin^2(1^\circ) + \sin^2(2^\circ) + \sin^2(3^\circ) + \cdots + \sin^2(89^\circ) + \sin^2(90^\circ)$$

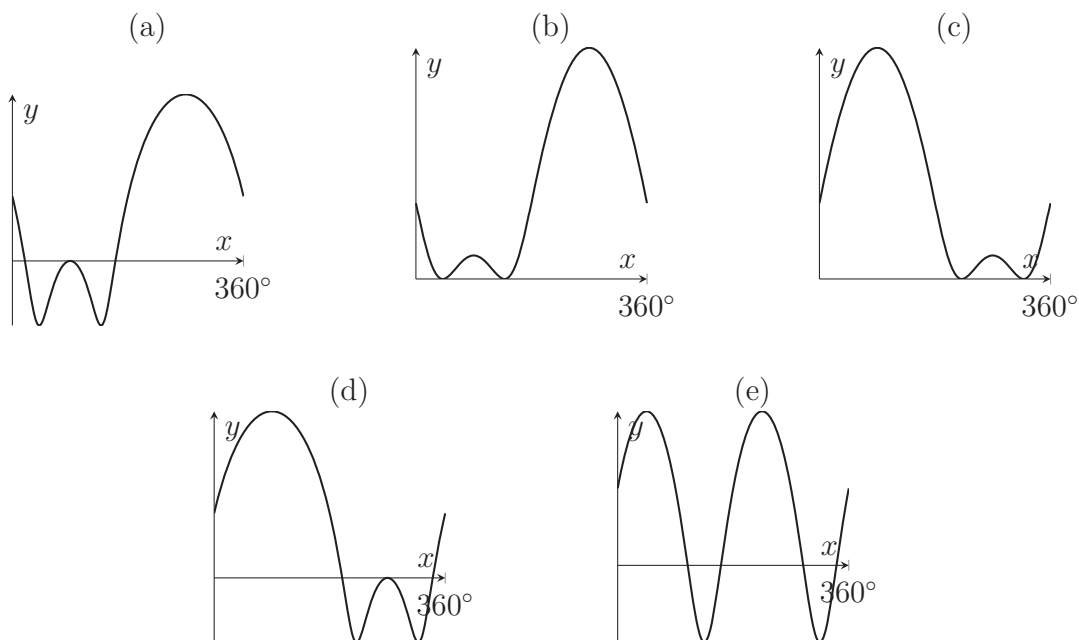
is equal to

- (a) 44, (b) $44\frac{1}{2}$, (c) 45, (d) $45\frac{1}{2}$, (e) 46.

H. Which of the following graphs shows

$$y = \log_2 (9 - 8 \sin x - 6 \cos^2 x)$$

in the range $0 \leq x \leq 360^\circ$?



I. A sequence is defined by $a_0 = 2$ and then for $n \geq 1$, a_n is one more than the product of all previous terms (so $a_1 = 3$ and $a_2 = 7$, for example). It follows that for all $n \geq 1$,

- (a) $a_n = 4a_{n-1} - 5$,
- (b) $a_n = a_{n-1}(a_{n-1} - 1) + 1$,
- (c) $a_n = 2a_{n-1}(a_{n-1} - 3) + 7$,
- (d) $a_n = \frac{3}{2}n^2 - \frac{1}{2}n + 2$,
- (e) None of the above.

J. Four distinct real numbers a , b , c , and d are used to define four points

$$A = (a, b), \quad B = (b, c), \quad C = (c, d), \quad D = (d, a).$$

The quadrilateral $ABCD$ has all four sides the same length

- (a) if and only if $(a - b)^2 = (c - d)^2$,
- (b) if and only if $(a - c)^2 = (b - d)^2$,
- (c) if and only if $(a - d)^2 = (b - c)^2$,
- (d) if and only if $a - b + c - d = 0$,
- (e) for no values of a , b , c , d .

Turn over

2. For ALL APPLICANTS.

In this question you may use without proof the following fact:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots - \frac{x^n}{n} \cdots \quad \text{for any } x \text{ with } |x| < 1.$$

[Note that $\ln x$ is alternative notation for $\log_e x$.]

- (i) By choosing a particular value of x with $|x| < 1$, show that

$$\ln 2 = \frac{1}{2} + \frac{1}{2 \times 2^2} + \frac{1}{3 \times 2^3} + \frac{1}{4 \times 2^4} + \frac{1}{5 \times 2^5} + \cdots$$

- (ii) Use part (i) and the fact that

$$\frac{1}{n2^n} < \frac{1}{3 \times 2^n} \quad \text{for } n \geq 4$$

to find the integer k such that $\frac{k}{24} < \ln 2 < \frac{k+1}{24}$.

- (iii) Show that

$$\ln \left(\frac{3}{2} \right) = \frac{1}{2} - \frac{1}{2 \times 2^2} + \frac{1}{3 \times 2^3} - \frac{1}{4 \times 2^4} + \frac{1}{5 \times 2^5} - \cdots$$

and deduce that

$$\ln 3 = 1 + \frac{1}{3 \times 2^2} + \frac{1}{5 \times 2^4} + \frac{1}{7 \times 2^6} + \cdots$$

- (iv) Deduce that $\frac{13}{12} < \ln 3 < \frac{11}{10}$.

- (v) Which is larger: 3^{17} or 4^{13} ? Without calculating either number, justify your answer.

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3.

For APPLICANTS IN $\left\{ \begin{array}{l} \text{MATHEMATICS} \\ \text{MATHEMATICS \& STATISTICS} \\ \text{MATHEMATICS \& PHILOSOPHY} \\ \text{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ ONLY.

Computer Science and *Computer Science \& Philosophy* applicants should turn to page 20.

The degree of a polynomial is the highest exponent that appears among its terms. For example, $2x^6 - 3x^2 + 1$ is a polynomial of degree 6.

- (i) A polynomial $p(x)$ has a turning point at $(0, 0)$. Explain why $p(0) = 0$ and why $p'(0) = 0$, and explain why there is a polynomial $q(x)$ such that

$$p(x) = x^2q(x). \quad (*)$$

- (ii) A polynomial $r(x)$ has a turning point at $(a, 0)$ for some real number a . Write down an expression for $r(x)$ that is of a similar form to the expression $(*)$ above. Justify your answer in terms of a transformation of a graph.
- (iii) You are now given that $f(x)$ is a polynomial of degree 4, and that it has two turning points at $(a, 0)$ and at $(-a, 0)$ for some positive number a .
- (a) Write down the most general possible expression for $f(x)$. Justify your answer.
- (b) Describe a symmetry of the graph of $f(x)$, and prove algebraically that $f(x)$ does have this symmetry.
- (c) Write down the x -coordinate of the third turning point of $f(x)$.
- (iv) Is there a polynomial of degree 4 which has turning points at $(0, 0)$, at $(1, 3)$, and at $(2, 0)$? Justify your answer.
- (v) Is there a polynomial of degree 4 which has turning points at $(1, 6)$, at $(2, 3)$, and at $(4, 6)$? Justify your answer.

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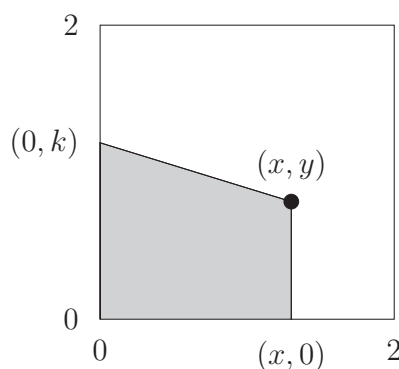
4.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \text{MATHEMATICS} \\ \text{MATHEMATICS \& STATISTICS} \\ \text{MATHEMATICS \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

Mathematics & Computer Science, Computer Science and Computer Science & Philosophy applicants should turn to page 20.

Charlie is trying to cut a cake. The cake is a square with side length 2, and its corners are at $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$. Charlie's first cut is a straight line segment from the point (x, y) to $(x, 0)$, where $0 \leq x \leq 2$ and $0 \leq y \leq 2$.

Charlie plans to make a second straight cut from the point (x, y) to a point $(0, k)$ somewhere on the left-hand edge of the cake. This will make a slice of cake which is bounded to the left of the first cut and bounded below the second cut.



- (i) Find the area of the slice of cake in terms of x , y , and k . Check your expression by verifying that if $x = 1$ and $y = 1$, then choosing $k = 1$ gives a slice of cake with area 1.
- (ii) Find another point (x, y) on the cake such that choosing $k = 1$ gives a slice of cake with area 1.
- (iii) Show that it is only possible to choose a value of k that gives a slice of cake with area 1 if both $xy \leq 2$ and $x(2 + y) \geq 2$.
- (iv) Sketch the region R of the cake for which both inequalities in part (iii) hold, indicating any relevant points on the edges of the cake.
- (v) Charlie may instead plan to make the second straight cut from (x, y) to a point $(m, 2)$ on the top edge of the cake in order to make a slice bounded to the left of the two cuts. Find two necessary and sufficient inequalities for x and y which must both hold in order for this to give a slice of area 1 for some value of m . Sketch the region of the cake for which both inequalities hold.

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5. For ALL APPLICANTS.

A *triangular triple* is a triple of positive integers (a, b, c) such that we can construct a triangle with sides of length a , b and c . This means that the sum of any two of the numbers is strictly greater than the third; so if $a \leq b \leq c$, then it is equivalent to requiring $a + b > c$. For example, $(3, 3, 3)$ and $(4, 5, 3)$ are triangular triples, but $(1, 3, 2)$ and $(3, 3, 6)$ are not. For any positive integer P , we define $f(P)$ to be the number of triangular triples such that the perimeter $a + b + c$ is equal to P . Triples with the same numbers, but in a different order, are counted as being distinct. So $f(12) = 10$, because there are 10 triangular triples with perimeter 12, shown below:

$(3, 4, 5)$	$(3, 5, 4)$	$(4, 3, 5)$	$(4, 5, 3)$	$(5, 3, 4)$	$(5, 4, 3)$
$(2, 5, 5)$	$(5, 2, 5)$	$(5, 5, 2)$			
$(4, 4, 4)$					

- (i) Write down the values of $f(3)$, $f(4)$, $f(5)$ and $f(6)$.
- (ii) If (a, b, c) is a triangular triple, show that $(a + 1, b + 1, c + 1)$ is also a triangular triple.
- (iii) If (x, y, z) is a triangular triple, with $x + y + z$ equal to an even number greater than or equal to 6, show that each of x, y, z is at least 2 and that $(x - 1, y - 1, z - 1)$ is also a triangular triple.
- (iv) Using the previous two parts, prove that for any positive integer $k \geq 3$,

$$f(2k - 3) = f(2k).$$

- (v) We will now consider the case where $P \geq 6$ is even, and we will write $P = 2S$.
 - (a) Show that in this case (a, b, c) is a triangular triple with $a + b + c = P$ if and only if each of a, b, c is strictly smaller than S .
 - (b) For any a such that $2 \leq a \leq S - 1$, show that the number of possible values of b such that $(a, b, P - a - b)$ is a triangular triple is $a - 1$. Hence find an expression for $f(P)$ for any even $P \geq 6$.
- (vi) Find $f(21)$.

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6.

For APPLICANTS IN $\left\{ \begin{array}{l} \text{COMPUTER SCIENCE} \\ \text{MATHEMATICS \& COMPUTER SCIENCE} \\ \text{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ ONLY.

Distinct numbers are arranged in an $m \times n$ rectangular table with m rows and n columns so that in each row the numbers are in increasing order (left to right), and in each column the numbers are in increasing order (top to bottom). Such a table is called a *sorted* table and each location of the table containing a number is called a *cell*. Two examples of sorted tables with 3 rows and 4 columns (and thus $3 \times 4 = 12$ cells) are shown below.

3	12	33	64
15	26	37	78
19	40	51	92

5	22	53	68
18	36	67	78
19	45	81	92

We index the cells of the table with a pair of integers (i, j) , with the top-left corner being $(1, 1)$ and the bottom-right corner being (m, n) . Observe that the smallest entry in a sorted table can only occur in cell $(1, 1)$; however, note that the second smallest entry can appear either in cell $(1, 2)$, as in the first example above, or in cell $(2, 1)$ as in the second example above.

(i) (a) Assuming that $m, n \geq 3$, where in an $m \times n$ sorted table can the third-smallest entry appear?

(b) For any $k \geq 4$ satisfying $m, n \geq k$, where in an $m \times n$ sorted table can the k^{th} smallest entry appear? Justify your answer.

(ii) Given an $m \times n$ sorted table, consider the problem of determining whether a particular number y appears in the table. Outline a procedure that inspects at most $m + n - 1$ cells in the table, and that correctly determines whether or not y appears in the table. Briefly justify why your procedure terminates correctly in no more than $m + n - 1$ steps.

[Hint: As the first step, consider inspecting the top-right cell.]

(iii) Consider an $m \times n$ table, say A , which might not be sorted; an example is shown below. Obtain table B from A by re-arranging the entries in each row so that they are in sorted order. Then obtain table C from B by re-arranging the entries in each column so that they are in sorted order. Fill in tables B and C here:

A:

33	92	46	24
25	26	37	8
49	40	81	22

→

B:

→

C:

(iv) Show that for *any* $m \times n$ table A , performing the two operations from part (iii) results in a sorted table C .

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7.

For APPLICANTS IN $\left\{ \begin{array}{l} \text{COMPUTER SCIENCE} \\ \text{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ ONLY.

Throughout this question, all functions will be Boolean functions of Boolean input variables. A Boolean variable can be either 0 or 1. A Boolean function may have one or more Boolean input variables, and the output of a Boolean function is also either 0 or 1. Three *elementary* Boolean functions are defined as follows:

- The function $\min(x_1, \dots, x_k)$ can take any number of inputs. It outputs the value 1 exactly when each of its inputs is 1, that is the output of the function is the *minimum* value among its inputs.
- The function $\max(x_1, \dots, x_k)$ can take any number of inputs. It outputs the value 1 exactly when at least one of its inputs is 1, that is the output of the function is the *maximum* value among its inputs.
- The function **flip** takes a single input and is defined as $\text{flip}(x_1) = 1 - x_1$.

First we will consider Boolean functions obtained by combining the three elementary Boolean functions. One such function is shown below:

$$f(x_1, x_2, x_3) = \min(\max(x_1, x_2, x_3), \text{flip}(\min(x_1, x_2, x_3))).$$

- (i) Describe in words when the function f outputs 1 and when it outputs 0.
- (ii) The function **majority**(x_1, \dots, x_k) takes k inputs and outputs 1 exactly when strictly greater than $k/2$ of its inputs are 1. Explain how you could combine elementary Boolean functions to obtain the following functions:
 - (a) **majority**(x_1, x_2)
 - (b) **majority**(x_1, x_2, x_3)

Now we will consider Boolean functions that can be obtained by combining only **majority** functions.

- (iii) There are exactly 16 distinct Boolean functions of two input variables. Some of these can be represented using only **majority** functions that take 3 inputs; the use of 0 or 1 as fixed inputs to **majority** is permitted. For example, $\text{majority}(x_1, x_2, 1)$ represents the function $\max(x_1, x_2)$.

Find any four other Boolean functions of two variables that can be represented by combining one or more **majority** functions of 3 inputs. Write your answers in terms of **majority** functions.

- (iv) Give an example of a Boolean function g of two input variables that cannot be represented by combining **majority** functions (of any number of inputs). You should write your answer by explicitly specifying $g(0, 0)$, $g(0, 1)$, $g(1, 0)$ and $g(1, 1)$. Justify your answer.

In the last part, you may express Boolean functions by combining any of the elementary Boolean functions or the **majority** function.

- (v) Consider four input variables x_1, x_2, x_3, x_4 . Let $z_1 = \min(x_1, x_2)$, $z_2 = \min(x_2, x_3)$, $z_3 = \min(x_3, x_4)$, $z_4 = \min(x_4, x_1)$. It is sometimes possible to represent a function $s(x_1, x_2, x_3, x_4)$ using a function $t(z_1, z_2, z_3, z_4)$. For example, $\min(x_1, x_2, x_3, x_4) = \min(z_1, z_2, z_3, z_4)$, as both functions output 1 if and only if all four x_i are 1.

Can you represent the following functions of inputs x_1, x_2, x_3, x_4 as some Boolean function of inputs z_1, z_2, z_3, z_4 ? Justify your answers.

- (a) **majority**(x_1, x_2, x_3, x_4).
- (b) The function **parity**(x_1, x_2, x_3, x_4) which outputs 1 exactly when an odd number of its inputs are 1.

End of last question

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