ANDREW JOBBINGS SENIOR KANGAROO

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SOLUTIONS

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. Adil was born in 2015. His younger sister Bav was born in 2018. What is the minimum number of days by which Adil is older than Bav?

SOLUTION **732**

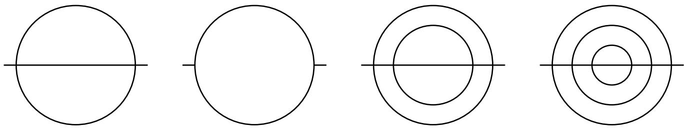
The closest the birth dates can be are 31st December 2015 and 1st January 2018. Since 2016 was a leap year, the number of days these are apart is $1 + 366 + 365 = 732$.

2. The total T is obtained as the sum of the integers from 2006 to 2036 inclusive. What is the sum of all the prime factors of T?

SOLUTION **121**

We can write the sum as $(2021 - 15) + (2021 - 14) + \ldots + (2021 - 1) + 2021 + (2021 + 1) + \ldots + (2021 + 14) + (2021 + 15) = 31 \times 2021 = 31 \times 43 \times 47$. The sum of all the prime factors of T is $31 + 43 + 47 = 121$.

3. How many of the figures shown can be drawn with one continuous line without drawing a segment twice?



SOLUTION **003**

All are possible except for the second figure. Each of the small lines must either start or end the continuous line. But then only one of the semicircles in the second figure can be included.

4. On each side of a right-angled triangle, a semicircle is drawn with that side as a diameter. The areas of the three semicircles are x^2, $3x$ and 180 where x^2 and $3x$ are both less than 180. What is the area of the smallest semicircle ?

SOLUTION **036**

The areas of a square of side $2r$ and its inscribed circle are in the ratio $4r^2 : \pi r^2 = 4 : \pi$. Therefore, the area of the square on each side of the triangle and the area of the corresponding semicircle are in the ratio $4 : \frac{\pi}{2} = 8 : \pi$. This is independent of side length and we therefore may use the Pythagoras' equation on the areas of the semicircles. This gives $x^2 + 3x = 180$. The solutions to this equation are $x = -15$ (which is impossible since this would give an area of $3 \times -15 = -45$) and $x = 12$, meaning the required area is 36.

5. $T = \sqrt{(2021 + 2021) + (2021 - 2021) + (2021 \times 2021) + (2021 \div 2021)}$.
What is the largest prime factor of T?

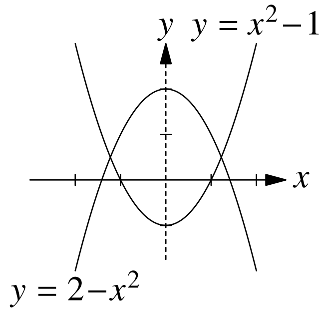
SOLUTION **337**

Let $x = 2021$. Then $T = \sqrt{2x + x^2 + 1} = x + 1 = 2022$.
Since $2022 = 2 \times 3 \times 337$, the largest prime factor of T is 337.

6. Into how many regions do the x-axis and the graphs of $y = 2 - x^2$ and $y = x^2 - 1$ split the plane?

SOLUTION **010**

As can be seen in the diagram the two graphs and the x-axis split the plane into ten regions.



7. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.

101	102	103	104	105

On the reverse, each card has one of five different positive integers: a, b, c, d and e respectively.
We know that $c = be$, $a + b = d$ and $e - d = a$.
Frankie picks up the card which has the largest integer on its reverse. What number is on the front of Frankie's card?

SOLUTION **103**

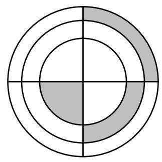
Since $e - d = a$ we have $e = a + d$. Therefore, $e > a$ and $e > d$.
Since $c = be$ we have $c > e$ and $c > b$, for if either of e, b equalled 1 then c would equal the other.
Therefore, $c > e > a$, $c > e > d$ and $c > b$.
The largest integer is c and the correct answer is 103.

8. The geometric mean of a set of n positive numbers is defined as the n-th root of the product of those numbers.
Yasmeen writes down a set of four numbers which have a geometric mean of 2048.
Zak writes down a set of four numbers which have a geometric mean of 8.
What is the geometric mean of the combined set of the eight numbers written by Yasmeen and Zak?

Solution **128**

Let Yasmeen's numbers be y_1, y_2, y_3 and y_4; and Zak's be z_1, z_2, z_3 and z_4.
We are given that $\sqrt[4]{y_1y_2y_3y_4} = 2048$ and $\sqrt[4]{z_1z_2z_3z_4} = 8$.
Therefore $\sqrt[4]{y_1y_2y_3y_4z_1z_2z_3z_4} = \sqrt[4]{y_1y_2y_3y_4} \times \sqrt[4]{z_1z_2z_3z_4} = 2048 \times 8 = 2^{11} \times 2^3 = 2^{14}$.
We now take the square-root of both sides getting $\sqrt[8]{x_1x_2x_3x_4y_1y_2y_3y_4} = 2^7 = 128$.

9. In the figure shown there are three concentric circles and two perpendicular diameters. The three shaded regions have equal area. The radius of the small circle is 2. The product of the three radii is Y.
What is the value of Y^2?



Solution **384**

We note that the central circle, the inner hoop and the outer hoop are also equal in area.
Let the radii of the middle circle and outer circle be a and b respectively.
The area of the central circle is 4π.
The area of the inner hoop is $\pi a^2 - 4\pi$. Therefore, $\pi a^2 - 4\pi = 4\pi$. This simplifies to $a^2 = 8$.
The area of the outer hoop is $\pi b^2 - \pi a^2 = \pi b^2 - 8\pi$.
Therefore, $\pi b^2 - 8\pi = 4\pi$. This simplifies to $b^2 = 12$.
The product of the radii, Y, is $2 \times a \times b$. Therefore, $Y^2 = 4 \times a^2 \times b^2 = 4 \times 8 \times 12 = 384$.

10. A dealer bought two cars. He sold the first one for 40% more than he paid for it and the second one for 60% more than he paid for it. The total sum he received for the two cars was 54% more that the total sum he paid for them. When written in its lowest terms, the ratio of the prices the dealer paid for the first and the second car was $a : b$.
What is the value of $11a + 13b$?

Solution **124**

Let the prices the dealer paid for the first and second cars be X and Y respectively. He sells the cars for $1.4X$ and $1.6Y$.
Since he received 54% more than he paid, we have $1.54(X + Y) = 1.4X + 1.6Y$.
Therefore, $0.14X = 0.06Y$ which gives us the ratio $X : Y = 6 : 14 = 3 : 7$.
Therefore $a = 3$ and $b = 7$ and $11a + 13b = 11 \times 3 + 13 \times 7 = 33 + 91 = 124$.

11. Billie has a die with the numbers 1, 2, 3, 4, 5 and 6 on its six faces.
Niles has a die which has the numbers 4, 4, 4, 5, 5 and 5 on its six faces.
When Billie and Niles roll their dice the one with the larger number wins. If the two numbers are equal it is a draw.
The probability that Niles wins, when written as a fraction in its lowest terms, is $\frac{p}{q}$.
What is the value of $7p + 11q$?

Solution **181**

Niles rolls a 4 with probability $\frac{1}{2}$, and then wins if Billie rolls a 1, 2 or 3, with probability $\frac{1}{2}$.
Niles rolls a 5 with probability $\frac{1}{2}$, and then wins if Billie rolls a 1, 2, 3 or 4, with probability $\frac{2}{3}$.
Therefore, the probability of Niles winning is $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{1}{4} + \frac{1}{3} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$.
Therefore $p = 7$, $q = 12$ and $7p + 11q = 7 \times 7 + 11 \times 12 = 181$.

12. There are 2021 balls in a crate. The balls are numbered from 1 to 2021. Erica works out the digit sum for each ball. For example, the digit sum of 2021 is 5, since $2 + 0 + 2 + 1 = 5$.
Erica notes that balls with equal digit sums have the same colour and balls with different digit sums have different colours.
How many different colours of balls are there in the crate?

Solution **028**

The largest possible digit sum is that of 1999 which is 28. The smallest is that of 1, which is 1. Each of the 9s in 1999 can be replaced by any of 0, 1, . . ., 8. So all digit sums between 1 and 28 can be achieved. Therefore there are 28 different digit sums and colours.

13. A multiplication table of the numbers 1 to 10 is shown. What is the sum of all the odd products in the complete table?

×	1	2	3	⋯	10
1	1	2	3	⋯	10
2	2	4	6	⋯	20
⋮	⋮				⋮
10	10	20	30	⋯	100

Solution **625**

The odd numbers in the table are all the products of two of the odd numbers 1, 3, 5, 7, 9.
So their sum is $(1 + 3 + 5 + 7 + 9)(1 + 3 + 5 + 7 + 9) = 25^2 = 625$.

14. The graph of $(x^2 + y^2 - 2x)^2 = 2(x^2 + y^2)^2$ meets the x-axis in p different places and meets the y-axis in q different places.
What is the value of $100p + 100q$?

SOLUTION **400**

The graph of $(x^2 + y^2 - 2x)^2 = 2(x^2 + y^2)^2$ meets the x-axis when $y = 0$.
i.e. $(x^2 - 2x)^2 = 2(x^2)^2$
i.e. $x^4 - 4x^3 + 4x^2 = 2x^4$
i.e. $0 = x^4 + 4x^3 - 4x^2$
i.e. $x = 0$ or $0 = x^2 + 4x - 4$
i.e. $x = 0$ or $x = -2 \pm 2\sqrt{2}$.
Therefore there are three points where the graph meets the x-axis, so $p = 3$.

The graph of $(x^2 + y^2 - 2x)^2 = 2(x^2 + y^2)^2$ meets the y-axis when $x = 0$,
i.e. $(y^2)^2 = 2(y^2)^2$
i.e. $y^4 = 2y^4$, with only one solution of $y = 0$.
Therefore there is only one point where the graph meets the y-axis, so $q = 1$.

The value of $100p + 100q$ is $100 \times 3 + 100 \times 1 = 400$.

15. Which is the lowest numbered statement which is true?
Statement 201: "Statement 203 is true".
Statement 202: "Statement 201 is true".
Statement 203: "Statement 206 is false".
Statement 204: "Statement 202 is false".
Statement 205: "None of the statements 201, 202, 203 or 204 are true".
Statement 206: "1 + 1 = 2".

SOLUTION **204**

Statement 206 is clearly true, so Statement 203 must be false. In turn Statement 201 must be false and therefore Statement 202 must also be false.
Statement 204 is therefore true. Therefore, Statement 204 is the lowest numbered true statement.

16. A polygon is said to be *friendly* if it is regular and it also has angles that when measured in degrees are either integers or *half-integers* (i.e. have a decimal part of exactly 0.5). How many different friendly polygons are there?

SOLUTION **028**

The interior angle will be an integer or half-integer precisely when the exterior angle is an integer or a half-integer respectively. A regular polygon of n sides has exterior angle $\frac{360}{n}$. This is either an integer or a half-integer if, and only if, $\frac{720}{n}$ is an integer. Therefore, n must be a factor of 720. The factors of 720 are 720, 360, 240, 180, 144, 120, 90, 80, 72, 60, 48, 45, 40, 36, 30, 24, 20, 18, 16, 15, 12, 10, 9, 8, 6, 5, 4 and 3 (ignoring 2 and 1 neither of which is a valid number of sides for a polygon). There are 28 numbers in this list, so there are 28 different friendly polygons.

17. Find the value of R, given that the numbers Q and R are defined as:

$Q = 202^1 + 20^{21} + 2^{021}$;
R is the remainder when Q is divided by 1000.

SOLUTION **354**

The term 20^{21} is divisible by 10^{21} and therefore is divisible by 1000, meaning it will not contribute to the remainder on division by 1000.
The term $2^{021} = 2^{21} = 2^{10} \times 2^{10} \times 2 = 1024 \times 1024 \times 2 = (1000 + 24) \times (1000 + 24) \times 2$.
When multiplied out, all terms involving a 1000 can be ignored.
So this contributes $24 \times 24 \times 2 = 1152$, or equivalently 152, to the remainder.
Hence the total remainder is $R = 202^1 + 152 = 354$.

18. Next year, 2022, has the property that it may be written using at most two different digits, namely 2 and 0. How many such years will there be between 1 and 9999 inclusive?

Solution **927**

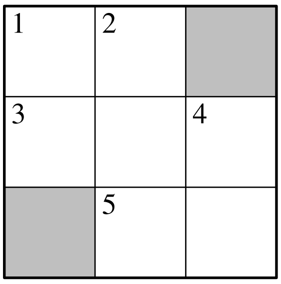
All of the 99 years 1 to 99 may be written using at most two different digits.
Three-digit years (i.e. those between 100 and 999 inclusive) can be written using only one different digit if they are of the form 'aaa', for example 111 or 222. There are 9 such years.
Three-digit years can be written using exactly two different digits a and b if they have any of the three following forms: 'aab', 'aba' or 'abb', where a,b are distinct digits and $a \neq 0$. There are 9 possibilities for a and for each there are then 9 possibilities for b. Therefore there are $9 \times 9 \times 3 = 243$ such years.
Four-digit years (i.e. those between 1000 and 9999 inclusive) can be written using only one different digit if they are of the form 'aaaa', for example 3333 or 4444. There are 9 such years.
Four-digit years can be written using exactly two different digits a and b if they have any of the seven following forms: 'aaab', 'aaba', 'abaa', 'aabb', 'abab', 'abba' or 'abbb', where a,b are distinct digits and $a \neq 0$. There are 9 possibilities for a and for each there are then 9 possibilities for b. Therefore there are $9 \times 9 \times 7 = 567$ such years.
Therefore, the total number of such years is $99 + 9 + 243 + 9 + 567 = 927$.

19. The function $f(x)$ is defined as $f(x) = \frac{x-1}{x+1}$.
The equation $f(x^2) \times f(x) = 0.72$ has two solutions a and b, where $a > b$.
What is the value of $19a + 7b$?

Solution **134**

We are required to solve $\frac{x^2-1}{x^2+1} \times \frac{x-1}{x+1} = \frac{18}{25}$, which simplifies to $\frac{(x-1)(x-1)}{x^2+1} = \frac{18}{25}$.
This leads to the quadratic equation $7x^2 - 50x + 7 = 0$ which factorises as $(7x - 1)(x - 7) = 0$.
Therefore, $a = 7$ and $b = \frac{1}{7}$ and $19a + 7b = 19 \times 7 + 7 \times \frac{1}{7} = 133 + 1 = 134$.

20. Each cell in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct. What is the answer to 2 DOWN?



ACROSS

1. A prime which is the sum of two squares
3. Twice the answer to 2 DOWN

DOWN

1. $p \times q$, where p, q are prime and $q = p + 4$
4. 60% of 5 ACROSS

SOLUTION **397**

The only two-digit pq with $q = p + 4$ are $77 = 7 \times 11$ and $21 = 3 \times 7$. The latter of these is not possible, since 3 ACROSS cannot begin with a 1 as it is twice a three-digit number.
So the answer to 1 DOWN is 77.
We now know the first digit of 1 ACROSS is 7, meaning that as primes 71, 73 and 79 are possible. Of these, the only one that can be expressed as the sum of two squares is $73 = 64 + 9$.
The clue for 4 DOWN tells us that 5× 4 DOWN = 3× 5 ACROSS. Hence 5 ACROSS is a multiple of 5 and 4 DOWN is a multiple of 3. The first of these shows that the bottom right entry is 5 (since 0 is not allowed); and the second then shows that 4 DOWN is a multiple of 15.
By the clue for 3 ACROSS, the first digit of 4 DOWN is even. It cannot be 0, 2, 6, or 8 because 4 DOWN is a multiple of 3. Therefore 4 DOWN = 45 and 5 ACROSS = 75.
Finally, given the first digits of 2 DOWN and 3 ACROSS and the clue for 3 ACROSS, we see that the middle digit must be at least 5 and we then can check quickly that 9 is the only possible value.
Hence 397 is the solution.