SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 27th November 2015

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

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1. 180 The number of gold coins in the original pile is $0.02 \times 200 = 4$. These form 20% of the final pile. Therefore there are $4 \times 5 = 20$ coins left. Hence the number of silver coins Simon removes is $200 - 20 = 180$.

2. 28 The expression can be simplified in stages as follows:

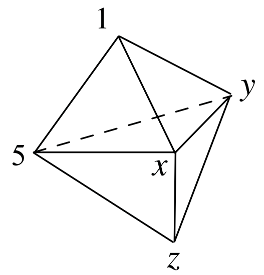
$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5}}} = 1 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{6}{5}\right)}} = 1 + \cfrac{1}{1 + \cfrac{5}{6}} = 1 + \cfrac{1}{\left(\frac{11}{6}\right)} = 1 + \frac{6}{11} = \frac{17}{11} = \frac{a}{b}.$$

Hence the value of $a + b$ is $17 + 11 = 28$.

3. 11 Let the three missing integers be x, y and z, as shown. Consider the 'top' three faces. Since the sum of the three numbers at the vertices of each face is the same, we have

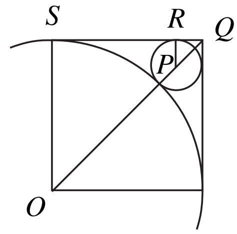
$$1 + 5 + x = 1 + x + y = 1 + 5 + y$$

and hence $x = y = 5$. Therefore the sum of the numbers on a face is equal to $5 + 5 + 1 = 11$. But $x + y + z$ is equal to the sum of the numbers on a face. Hence the sum of the other three numbers that Andrew will write is 11.



4. 7 The first six socks Rachel takes out could consist of three different coloured socks and the three socks with holes in, in which case she would not have a pair of socks the same colour without holes in. However, whatever colour her next sock is, she must then complete a pair. Hence she must take seven socks to be certain of getting a pair of socks the same colour without holes in.

5. **50** Let O and P be the centres of the large and small circles respectively and label points Q and S as shown in the diagram. Let the radius of the small circle be r cm. Draw line PR so that R is on QS and PR is parallel to OS. Draw in line OQ. Since triangle OQS is right-angled and isosceles, $OQ^2 = 10^2 + 10^2$ by Pythagoras. Hence $OQ = 10\sqrt{2}$ cm. Similarly, since triangle PQR is right-angled and isosceles, $PQ = r\sqrt{2}$ cm. Note that angle OQS = angle $PQS = 45°$ so OPQ is a straight line. Therefore $10\sqrt{2} = 10 + r + r\sqrt{2}$. This has solution

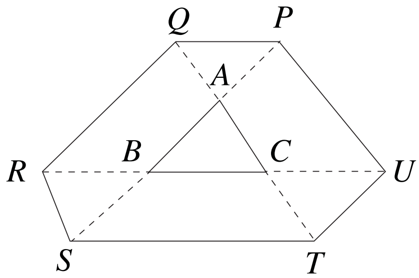


$$r = \frac{10(\sqrt{2}-1)}{\sqrt{2}+1} = \frac{10(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{10(2+1-2\sqrt{2})}{2-1} = 30 - 20\sqrt{2}.$$

Hence the radius of the small circle is $(30 - 20\sqrt{2})$ cm and the value of $a + b$ is $30 + 20 = 50$.

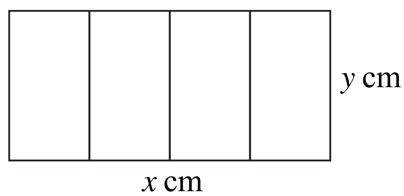
6. **7** Let the five integers be p, q, r, s and t with $p \leqslant q \leqslant r \leqslant s \leqslant t$. The median of the list is r and, since the mode is one less than the median, $p = q = r - 1$ and $r < s < t$. The mean is one more than the median and hence the total of the five integers is $5(r + 1)$. Therefore $r - 1 + r - 1 + r + s + t = 5r + 5$ and hence $s + t = 2r + 7$. Since the smallest possible value of s is $r + 1$, the maximum value of t is $r + 6$. Hence the largest possible value of the range of the five integers is $r + 6 - (r - 1) = 7$.

7. **156** Consider triangles ABC and AST. Angles CAB and TAS are equal because they are the same angle, $SA = 2BA$ and $TA = 2CA$. Hence triangles ABC and AST are similar. The ratio of their sides is 1 : 2 and hence the ratio of their areas is $1^2 : 2^2 = 1 : 4$. Therefore the area of triangle AST is $4 \times 12\ \text{cm}^2 = 48\ \text{cm}^2$ and hence the area of $BSTC$ is $(48 - 12)\ \text{cm}^2 = 36\ \text{cm}^2$. In a similar way, it can be shown that each of the areas of $CUPA$ and $AQRB$ is also $36\ \text{cm}^2$. Next consider triangles ABC and APQ. Angles BAC and PAQ are equal using vertically opposite angles, $AB = AP$ and $AC = AQ$. Hence triangles ABC and APQ are congruent (SAS) and so the area of triangle APQ is $12\ \text{cm}^2$. In a similar way, it can be shown that the each of areas of triangles BRS and CTU is also $12\ \text{cm}^2$. Hence the total area of hexagon $PQRSTU$ in cm^2 is $(3 \times 36 + 4 \times 12) = 156$.



8. **672** The first grey kangaroo has only one red kangaroo smaller than itself. Apart from that, each grey kangaroo can be grouped with two red kangaroos whose heights lie between its height and that of the previous grey kangaroo. The number of such groups is $(2015 - 2)/3 = 671$. Hence there are 672 grey kangaroos in the mob.

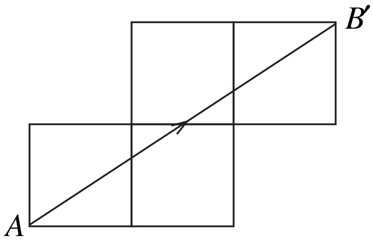
9. **28** Let the length of the original rectangle be x metres and let the height be y metres. Without losing any generality, assume the rectangle is sliced parallel to the height, as shown.



The information in the question tells us that $2x + 2y = 2\left(\frac{x}{4}\right) + 2y + 18$ and that $xy = \left(\frac{x}{4}\right)y + 18$. From the first equation, we have $\frac{3x}{2} = 18$ which has solution $x = 12$. Substitute this value into the second equation to obtain $12y = 3y + 18$, which has solution $y = 2$. Hence the perimeter of the large rectangle in metres is $2 \times 12 + 2 \times 2 = 28$.

10. 288 Katherine catches James after 8 minutes when she has jogged $\frac{8}{3}$ laps. In that time, James will have jogged one lap fewer so will have jogged $\frac{5}{3}$ laps. Therefore, James jogs $\frac{5}{3}$ laps in 8 minutes which is the same as 480 seconds. Hence he will jog $\frac{1}{3}$ of a lap in 96 seconds and so he jogs a whole lap in 288 seconds.

11. 52 A solution can be obtained by reflecting the square repeatedly in the cushion the ball strikes. The path of the ball is then represented by the line AB' in the diagram.



The length of the path can be calculated using Pythagoras Theorem. We have $(AB')^2 = (3 \times 2)^2 + (2 \times 2)^2$. Therefore $(AB')^2 = 36 + 16 = 52$ and so $AB' = \sqrt{52}$ metres and hence the value of k is 52.

12. 35 Chris's time for the ride when he rode at an average speed of x km/h was $\dfrac{210}{x}$ hours. His planned speed was $(x - 5)$ km/h when his time would have been $\dfrac{210}{x-5}$ hours. The question tells us that he completed the ride 1 hour earlier than planned, so $\dfrac{210}{x-5} - \dfrac{210}{x} = 1$.
Therefore $210x - 210(x - 5) = x(x - 5)$ and hence $1050 = x^2 - 5x$. Thus $x^2 - 5x - 1050 = 0$ and hence $(x - 35)(x + 30) = 0$. Therefore, since x is positive, $x = 35$.

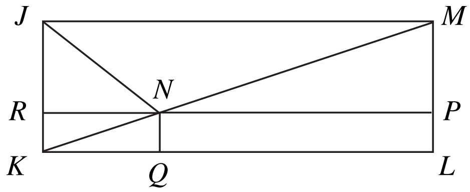
13. 13 Assume the man at the front of the queue is telling the truth and that everyone behind him always lies. However, then the person in third place in the queue would be telling the truth when he says that the person in second place always lies. This contradicts the original assumption and so the man at the front of the queue is lying. In this case, the man in second place is telling the truth, the man in third place is lying etc. Hence, every other person, starting with the first, is lying and so there are $1 + \frac{1}{2} \times 24 = 13$ people in the queue who always lie.

14. 30 The smallest number of contestants solving all four problems correctly occurs when the contestants who fail to solve individual problems are all distinct. In that case, the number failing to solve some question is $10 + 15 + 20 + 25 = 70$ and the number solving them all is $100 - 70 = 30$.

15. 14 First note that $165 = 3 \times 5 \times 11$. Hence, for '$XX4XY$' to be exactly divisible by 165, it must be exactly divisible by 3, 5 and 11. A number is divisible by 3 if and only if the sum of its digits is divisible by 3 so $3X + 4 + Y$ is divisible by 3 and hence $4 + Y$ is divisible by 3. A number is divisible by 5 if and only if its last digit is 5 or 0 so $Y = 5$ or 0. Since $4 + Y$ is divisible by 3 then $Y = 5$. A number is divisible by 11 if and only if the sum of its digits with alternating signs is divisible by 11 so $X - X + 4 - X + Y$ is divisible by 11. Hence $9 - X$ is divisible by 11 and so $X = 9$. Hence the value of $X + Y$ is $9 + 5 = 14$.

16. 64 Since adjacent digits differ by 1, each time the number has a digit that is a 1 or a 3, there is only one choice for the next digit as it must be a 2 whereas each time the number has a digit that is a 2, there are two choices for the next digit, namely 1 or 3. Consider all 10-digit numbers starting in a 1. There is only one choice for the second digit since it must be a 2, then two choices for the third digit, then one for the fourth etc. Altogether there are $1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1 = 16$ such numbers. Similarly there are 16 such numbers starting in 3. However, if we consider numbers starting in 2, there are two choices for the second digit then only one choice for the third then two for the fourth etc. Altogether there are $2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 2 = 32$ such numbers. Hence there are $16 + 16 + 32 = 64$ such numbers with the required property.

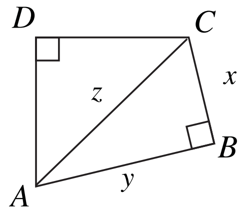
17. **16**



Let points P and Q be the points where the perpendiculars from N to ML and KL meet the lines and extend line PN so it meets JK at R, as shown in the diagram. Since JN is the bisector of angle MJK, angle $NJR = 45°$. Since angle JRN is 90°, triangle JRN is isosceles and $JR = RN$. Let the length of RN be x cm. Hence the lengths of JR and PM are also x cm. Observe that triangles NKQ and MNP are similar since they have the same angles. Therefore $\frac{1}{x} = \frac{x}{8}$ and so $x = \sqrt{8}$ since x is positive. The length of KL is equal to the sum of the lengths of NP and NR. Therefore, the length of KL is $(8 + \sqrt{8})$ cm. Hence, the value of $a + b$ is 16.

18. **2** Consider the equation $\frac{a}{b+c} = \frac{b}{c+a}$. Multiply each side by $(b + c)(c + a)$ to get $a^2 + ac = b^2 + bc$ and so $a^2 - b^2 + ac - bc = 0$. Therefore $(a - b)(a + b + c) = 0$. Hence $a = b$ or $a + b + c = 0$. Similarly, if we consider the equations $\frac{b}{c+a} = \frac{c}{a+b}$ and $\frac{c}{a+b} = \frac{a}{b+c}$, then $b = c$ or $a + b + c = 0$ and $c = a$ or $a + b + c = 0$ respectively. Therefore, the possible values of k when all three equations are satisfied simultaneously occur when $a = b = c$, giving $k = \frac{1}{2}$, or when $a + b + c = 0$, giving $k = -1$. Hence there are two possible values of k.

19. **100** Let the lengths of BC, AB and AC be x, y and z centimetres respectively. Let the area of $\triangle ACD$ be U cm^2 and let the area of $\triangle ABC$ be V cm^2. Note that $\triangle ACD$ is one quarter of the square which has AC as an edge. Hence $U = \frac{1}{4}z^2$. Next, using Pythagoras, $z^2 = x^2 + y^2 = (x + y)^2 - 2xy = 20^2 - 4V$. Hence $U = \frac{1}{4}(400 - 4V) = 100 - V$. Therefore the area in cm^2 of $ABCD$ is $U + V = 100$.



(Note: *Since the answer to the problem is independent of x and y, one could observe that the given properties of quadrilateral ABCD are satisfied by a square of side* 10 cm *which has area* 100 cm^2 *and conclude that this is therefore the required answer.*)

20. **247** First factorise N twice using the difference of two squares i.e. $N = 3^{16} - 1 = (3^8 - 1)(3^8 + 1) = (3^4 - 1)(3^4 + 1)(3^8 + 1) = 80 \times 82 \times (3^8 + 1)$. This shows that both 80 and 82 are divisors of N in the required range. The question tells us that 193 is a divisor of N and, since 193 is prime, it must be a divisor of $3^8 + 1 = 81 \times 81 + 1 = 6562$. Now observe that $6562 = 2 \times 3281$ and that $3281 \div 193 = 17$. Therefore $N = 80 \times 82 \times 2 \times 17 \times 193$ or $N = (2^4 \times 5) \times (2 \times 41) \times 2 \times 17 \times 193$.

Next consider the integers from 75 to 85 inclusive to see which could be divisors of N. Because N has no prime factors of 3 or 7, we know that 75, 77, 78, 81 and 84 are not divisors of N while the initial argument established that 80 and 82 are divisors of N. Both 79 and 83 are prime and $76 = 4 \times 19$ so, since N does not have a prime factor of 79, 83 or 19, these must also be excluded. This only leaves 85 to be considered. Note that $85 = 5 \times 17$ and both 5 and 17 are prime factors of N so 85 is a divisor of N. Hence the divisors of N in the required range are 80, 82 and 85 with sum 247.