SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 2nd December 2016

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

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1. 121 The sum $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21$ has eleven terms. Therefore the value of the required sum is $11 \times 11 = 121$.

2. 950

1	2	3	4	5	6	7	8	9	10	1050
11	12	13	14	15	16	17	18	19	20	*

We observe that in all but the rightmost column the value in the second row is ten larger than the value in the first row. There are 10 such columns. Therefore the sum of the leftmost ten elements of the second row is 100 more than the corresponding sum in the first row. To achieve the same total in each row, * will need to be 100 less than the value above it. Therefore * = 950.

3. 27 We first observe that any pair of cubes are mathematically similar. These cubes' surface areas are in the ratio 1:9, so that their lengths are in ratio 1:3 and that their volumes are in ratio 1:27.

Therefore Andrew may fill the larger cube in 27 visits, provided the smaller cube is completely filled on each occasion.

4. 125 The number will be of the form '$abcd$' where a, b and c are any odd digits and $d = 5$. Hence there are 5, 5, 5 and 1 possibilities for a, b, c and d respectively. Therefore there are $5 \times 5 \times 5 \times 1 = 125$ such numbers.

5. 9 The enclosed area is a concave quadrilateral with vertices at $(-3, 3)$, $(0, 0)$, $(3, 3)$ and $(0, -3)$. Considering this as two conjoined congruent triangles we find the area as $2 \times \frac{1}{2} \times 3 \times 3 = 9$.

6. 160 The tangent-radius property gives $\angle PSO = \angle PTO = 90°$. From the angle sum of quadrilateral $PTOS$ we may conclude that $\angle TPS = 30°$ and therefore that $\angle TPY = 20°$. By considering the angle sum of triangle PTY we conclude that the required total is $160°$.

7. **315** Suppose that the cross-section of the prism is an N-gon with N edges. The prism will have N edges in each of its 'end' faces and a further N edges connecting corresponding vertices of the end faces. Therefore the number of edges is $3N$ and hence is a multiple of 3. The only multiples of 3 in the given range are 312, 315 and 318. Since we know the total is odd, the prism has 315 edges.

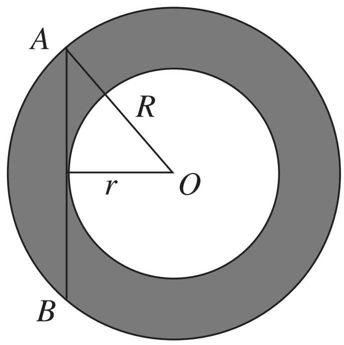
8. **7** Since squares of real numbers are non-negative, the sum can only be 0 if each expression in brackets is zero. Therefore the solutions of the equation are $x = \pm 3$, $y = \pm 2$ and $z = \pm 1$. We observe that the maximum and minimum values for $x + y + z$ are 6 and -6, and that since $x + y + z$ is the sum of one even and two odd numbers, that $x + y + z$ itself will be even.

It suffices to show that each even total between +6 and −6 can be attained.

$(+3) + (+2) + (+1) = +6$ $\quad$ $(+3) + (+2) + (-1) = +4$
$(+3) + (-2) + (+1) = +2$ $\quad$ $(+3) + (-2) + (-1) = 0$
$(-3) + (+2) + (-1) = -2$ $\quad$ $(-3) + (-2) + (+1) = -4$
$(-3) + (-2) + (-1) = -6$

Hence there are seven possible values for $x + y + z$.

9. **256** Let the radii of the larger and smaller circles be R and r respectively. Draw radius OA of the larger circle and drop the perpendicular from O to AB. By the tangent-radius property this perpendicular will be a radius of the smaller circle.



Now the area of the shaded region = area of larger circle − area of smaller circle.

The area of the shaded region = $\pi R^2 - \pi r^2 = \pi\left(R^2 - r^2\right)$.

But $R^2 - r^2 = 16^2 = 256$ (by Pythagoras' theorem), hence the area of the shaded region = 256π and therefore $k = 256$.

10. **103** Note that $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$. We observe that if any multiplication sign, other than the first, is replaced by an addition sign then each remaining product is at most 360. Therefore we retain each multiplication sign except the first which may be replaced by an addition sign to obtain a maximal value of 721.

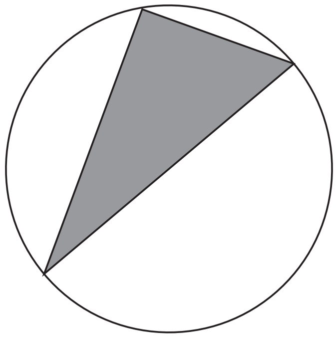
The prime factors of 721 are 7 and 103, of which 103 is the largest.

11. **183** We first observe that exactly one odd year and exactly one even year are under consideration.

In an odd year we need only consider odd months. January, March, May and July each has 16 odd days while September and November has 15. Therefore the number of days Stephanie will swim in the odd year is $4 \times 16 + 2 \times 15 = 94$.

In an even year we need only consider even months. April, June, August, October and December has 15 even days and February has 14 (regardless of whether or not it is a leap year). Therefore the number of days Stephanie will swim in the even year is $5 \times 15 + 14 = 89$. Hence she will swim for $94 + 89 = 183$ days over the two years.

12. 144 The regular 18-gon has a circumcircle, that is, a circle passing through all of its vertices. This is also the circumcircle of each right-angled triangle formed. In order for one of these triangle's angles to be a right angle, the opposite side needs to be a diameter, There are 9 possible choices of diameter. For each choice of diameter, there are 8 vertices on each side for the right angle, making 16 choices overall. For each choice of diameter there are 16 choices for the third vertex of the right-angled triangle.



13. 143 For a year Y to be expressible as as the sum of two positive integers p and q where $2p = 5q$ we require $p + q = Y$ and $2p = 5q$. From the first of these, it follows that $2p + 2q = 2Y$ and hence $5q + 2q = 2Y$.

Therefore $7q = 2Y$ from which it follows that Y is also divisible by 7 (since 2 and 7 are coprime).

We observe that $q = \frac{2Y}{7}$ will be an integer less than Y for all Y that are multiples of 7.

Then $p = Y - q$ will also be an integer.

We now must count all the multiples of 7 between 2000 and 3000 inclusive.

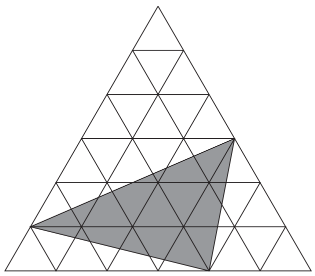
Since $1995 = 285 \times 7$ and $2996 = 428 \times 7$ there are $428 - 285 = 143$ multiples of 7 between 2000 and 3000 and hence there are 143 such years.

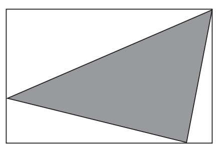
14. 35 Assume, without loss of generality, that $x \leqslant y$. Since x, y are positive integers and $xy = 105$, the possible values of (x, y) are (1, 105), (3, 35), (5, 21), (7, 15).

Since we require $13 + x > y$ for the triangle to exist, we may eliminate the first three of these possibilities, leaving only (7, 15) and conclude that the perimeter is $13 + 7 + 15 = 35$.

15. 110 For each small equilateral triangle, let the length of each side be x and the perpendicular height be h.

We may trap the shaded triangle in a rectangle as shown, where one vertex is coincident with one of the vertices of the rectangle and the other two vertices lie on sides of the rectangle.





The rectangle has width $4x$ and height $3h$. Therefore the rectangle's area is $12xh$.

The three additional (unshaded) right-angled triangles in the rectangle have areas $\frac{1}{2} \times 4x \times 2h = 4xh$, $\frac{1}{2} \times \frac{1}{2}x \times 3h = \frac{3}{4}xh$ and $\frac{1}{2} \times \frac{7}{2}x \times h = \frac{7}{4}xh$.

Therefore their total area is $4xh + \frac{3}{4}xh + \frac{7}{4}xh = \frac{13}{2}xh$.

Therefore $K = 12xh - \frac{13}{2}xh = \frac{11}{2}xh$.

Each of the 36 smaller equilateral triangles has area $\frac{1}{2}xh$ so we know that $\frac{1}{2}xh = 10$ and therefore that $xh = 20$.

Therefore $K = \frac{11}{2} \times 20 = 110$.

16. 87 The function $f(x)$ has the property that $3f(x) + 7f\left(\frac{2016}{x}\right) = 2x$. First observe that $\frac{2016}{8} = 252$. Therefore $3f(8) + 7f(252) = 16$ and $3f(252) + 7f(8) = 2 \times 252$. Let $f(8) = V$ and $f(252) = W$. Therefore $3V + 7W = 16$ and $3W + 7V = 504$. When these equations are solved simultaneously, we obtain $V = 87$ and $W = -35$ so that $f(8) = 87$.

17. 649 We observe that the total of all integers on the board at the start of the process is $10(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = 550$.

On each turn this total is increased by 1. Since we start with one hundred integers on the board and at each turn this number of integers is decreased by one, then 99 turns will be required to complete the process. Therefore the total of all integers on the board will increase by 99 over the course of the process. Hence the remaining number will be $550 + 99 = 649$.

18. 441 Let the smallest number be x. Therefore $x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 = (x+4)^2 + (x+5)^2 + (x+6)^2$ and hence

$x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 + x^2 + 6x + 9 = x^2 + 8x + 16 + x^2 + 10x + 25 + x^2 + 12x + 36$.

This can be rewritten as $4x^2 + 12x + 14 = 3x^2 + 30x + 77$ or $x^2 - 18x - 63 = 0$. Hence $(x - 21)(x + 3) = 0$, which has solutions $x = 21$ and $x = -3$. The question tells us that x is positive and therefore $x = 21$.

The square of the smallest of these integers is therefore $21^2 = 441$.

19. 421 When an odd number is subtracted from an odd square, an even (and hence composite) number is obtained. Similarly, when a multiple of 3 (or 7) is subtracted from a square of a multiple of 3 (or 7), a multiple of 3 (or 7) is obtained which is also composite. Therefore we need only consider three-digit squares that are neither odd nor a multiple of 3 (or 7). Hence the only squares we need to consider are $16^2 = 256$, $20^2 = 400$, $22^2 = 484$ and $26^2 = 676$ which yield differences of 235, 379, 463 and 655 respectively. It is easy to see that 235 and 655 are multiples of 5 and hence composite. Therefore only 379 and 463 remain as possible primes satisfying the given condition. After checking divisibilty by 11, 13, 17 and 19 for both, both are indeed seen to be prime and their mean is 421.

20. 116 Any code will start with a black strip and a white strip followed by a shorter barcode. Let $C(m)$ be the number of distinct barcodes of width m.

Those codes which start with BW will be followed by a code of width $m - 2$; so there will be $C(m - 2)$ of these. Likewise, there will be $C(m - 3)$ codes starting BBW, the same number starting BWW, and $C(m - 4)$ starting BBWW; and that exhausts the possibilites. So it follows that $C(m) = C(m-2) + 2C(m-3) + C(m-4)$.

When $m \leqslant 4$, it is simple to list all possible barcodes; namely B, BB, BWB and BBWB, BWBB, BWWB. Therefore $C(1) = C(2) = C(3) = 1$ and $C(4) = 3$. We can now calculate $C(m)$ for $m > 4$.

Thus $C(5) = C(3) + 2C(2) + C(1) = 1 + 2 + 1 = 4$, and continuing like this, we get $C(6) = 6$, $C(7) = 11$, $C(8) = 17$, $C(9) = 27$, $C(10) = 45$, $C(11) = 72$, $C(12) = 116$.