Cambridge Assessment Admissions Testing

STEP Hints and Solutions 2017

Mathematics

STEP 9465/9470/9475

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Contents

STEP Mathematics (9465, 9470, 9475)

STEP I 2017
Hints and Solutions

Question 1

Part (i)

Using $\tan x = \frac{\sin x}{\cos x}$ and rearranging will give an algebraic fraction where the denominator is u and the numerator is a multiple of $\frac{\mathrm{d}u}{\mathrm{d}x}$, after which the integration can be completed easily.

In a similar way, substituting $\cot x = \frac{\cos x}{\sin x}$ and rearranging will show that

$u = x\cos x - \sin x$ is a sensible choice of substitution to complete the second integration.

Part (ii)

For the first integration, differentiating the denominator of the fraction will show that an appropriate substitution is $u = x\sec^2 x - \tan x$, following which the integration follows easily. The second integration requires some rearrangement of the algebraic fraction: division of the numerator and denominator by $\cos^4 x$ puts it into a form where it should be clear that the same substitution will work for this integration as well.

Question 2

Part (i)

Taking definite integrals of the two sides of the inequality with the lower limit as 1 and the upper limit as x leads directly to the required result in the case $x \geq 1$. To show the result in the case $0 < x \leq 1$, note that $\frac{1}{t} \geq 1$ holds for $0 < t \leq 1$ and perform definite integrals with the lower limit as x and the upper limit as 1.

Part (ii)

Following the same process as for part (i) will lead to the required result.

Part (iii)

Integrating (*) and rearranging leads to one half of the inequality and integrating (**) and rearranging leads to the other half of the inequality. In both cases take limits of 1 and y for the integrals and consider the cases $0 < y < 1$ and $y > 1$ separately.

Question 3

Using implicit differentiation on the equation of the curve gives $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$, which then leads easily to an expression for the gradient of the curve at the point P and so the equation of the tangent can be deduced easily from this point. The equation of the tangent at Q can then be written down as it simply requires changing p to q throughout.

A diagram of the situation described then allows a strategy for the areas of each triangle to be worked out: for RST, one side is vertical and so this length can be used as the base and so the required measurements are easily deduced from the coordinates of the points. There are many ways of deducing the area of OPQ, for example by adding horizontal lines through P and Q and then considering the triangle to be a trapezium with two triangles cut away from it.

Question 4

Part (i)

The function $1 + r + r^2$ is recognisable as a quadratic function and so can be sketched by completing the square to identify the location of its minimum point. Since we know that $|r| < 1$, we can regard this as the domain and find the range of the function to deduce the possible values of p.

In the case where $1 < p < 3$ it can be seen from the graph that there is only one possible value for r as the horizontal line for this value of p only intersects the graph once. Since S is defined in terms of r, the value of S must also be determined uniquely.

In the other case, $S = \frac{1}{1-r}$ can be rearranged to make r the subject and then substituted into the function for p to deduce the required equation.

Part (ii)

Considering a graph of the function for q will again determine the values of q that determine T uniquely and a similar approach by substituting will lead to the quadratic equation required for the last part of the question.

Question 5

A diagram is very helpful in understanding the description in the first paragraph. From such a diagram expressions for the width and height can be identified which will then lead to the formula for the area of the rectangle. Since Q and R are aligned vertically, they must have the same y-coordinate and this can be used to give an expression for s in terms of the other variables.

When performing the differentiation required next, remember that s is a function of x and differentiate the expression for s to find $\frac{\mathrm{d}s}{\mathrm{d}x}$. This can then be substituted into the expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$.

The greatest possible area can be found by setting $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ and application of trigonometric identities will allow the final result to be shown.

Question 6

Part (i)

Note that if $\mathrm{f}(x)$ does not take any negative values then the value of the integral must be positive (and similarly, if it does not take any positive values then the value of the integral must be negative). The result then follows from this.

Part (ii)

The integral that needs to be considered can be shown to be equal to 0 and so the result from part (i) implies that $(x-\alpha)^2\mathrm{g}(x)$ must have both positive and negative values in the interval. Since $(x-\alpha)^2 > 0$ it must also be the case that $\mathrm{g}(x)$ has both positive and negative values in the interval and so it must also take the value 0 somewhere (by the result that can be assumed in this question).

Substituting the form of $\mathrm{g}(x)$ into the three integrals given in (*) gives a set of simultaneous equations that can be solved to find an example of such a function and it is then straightforward to confirm that there is a 0 by showing that the endpoints of the interval give different signs for the value of $\mathrm{g}(x)$.

Part (iii)

To be able to apply the result from part (ii), it will need to be the case that $\mathrm{g}(x) = \mathrm{h}'(x)$. Once the three integrals have been shown to have values that are consistent with part (ii) the result follows immediately.

Question 7

Part (i)

Applying the cosine rule to the triangle CMA leads to the required result and the corresponding result for CL is easy to deduce.

Part (ii)

Applying the cosine rule to triangle CLM leads directly to the first result required. Since the formula is symmetric in a, b and c, it must be the case that all three sides of LMN are equal and so the triangle is equilateral. The final result of this part can be shown by expressing the area of LMN in terms of the length of one side.

Part (iii)

The equivalence of the conditions follows by applying the formula for $\cos(A - B)$ and rearranging.

The final deduction will follow by combining the results from parts (ii) and (iii).

Question 8

Following the standard procedure for proof by induction establishes the first result.

Part (i)

The required fact about the sequence b_n can be proven by induction, and then division of (*) by b_n^2 will show that as $n \to \infty$, c_n will approach the root of a quadratic equation and this equation can be solved to give the required result.

Part (ii)

By considering $c_{n+1} - c_n$ it can be shown that the sequence is decreasing. Hence c_n must therefore be greater than the limit of the sequence for all n. Rearranging this equation then leads to the required inequality.

Finally, calculating the first few terms of the sequences leads to the required approximations.

Question 9

Part (i)

By considering the horizontal speed it is possible to find the amount of time before the particle passes through P. Considering the vertical speed then allows a relationship between u and α to be deduced. Differentiating with respect to α and rearranging then leads to the first result.

The relation between α and β can then be deduced by considering the relationship between the graphs of $y = \tan x$ and $y = \cot x$.

Part (ii)

Considering the horizontal and vertical components of the velocity as the particle passes through P allows the tangent of the angle at which it is travelling to be deduced. Graphical considerations can then be used to deduce the relationship between this angle and α.

Question 10

Part (i)

Standard procedures considering conservation of momentum and the coefficient of restitution can be used to show the relationship between each of the velocities in terms of the variables required.

Part (ii)

It is clear that each particle must be involved in at least one collision. To show that there are no more than two collisions it is required to show that each particle will still be moving faster than the one behind it following the second collision.

Part (iii)

When $e = \lambda$ all of the particles will have the same velocity following the collision and it can be seen that the kinetic energies will form a geometric progression and so the sum to infinity can be calculated and then the fraction of kinetic energy lost deduced.

Part (iv)

In this case the particles can be seen to come to rest eventually and so all of the kinetic energy is lost.

Question 11

Adding the forces to the diagram and then resolving parallel and perpendicular to the slope gives two equations relating all of the forces and lengths. Taking moments about the centre of the rod gives a further equation. Solving these equations and using the relationship between frictional and reaction forces then allows an equation in terms of tangents of the appropriate variables to be found. Rearranging then allows the formula for $\tan(A+B)$ to be obtained and therefore the required result can be shown to be the only solution to the equation that matches the situation described.

Question 12

Part (i)

The probability that any one participant does not pick the winning number is $1-\frac{1}{N}$ and since the participants' choices are independent, the probability that no participant picks the winning number is this value raised to the power N. The profit is then £cN in the case where no one picks the winning number and £$(cN-J)$ if someone does pick the winning number. Applying the given approximations then allows the result to be shown.

Part (ii)

The first relationship that is needed follows from the fact that the probability that one of the numbers will be chosen must be 1. The expected profit can then be calculated by following a similar procedure as in part (i).

In the case described at the end of this part the information along with the relationship between a, b and γ is enough to deduce the values of a and b and that the organiser expects to make a profit if $2Nc=J$.

Question 13

It should be clear that the first slice of the loaf of bread can never be the second of two slices to make a sandwich and so the value of s_1 can be deduced. To explain the first of the two equations note that to use the r^{th} slice of bread for toast means that the previous slice must have been either the second slice of a sandwich or used to make toast (the total probability of this is $(s_{r-1} + t_{r-1})$). This must then be multiplied by the probability of using this next slice to make toast. Since it is not possible to follow this reasoning for the first slice of toast, the equation can only be valid for values of 2 or greater. Similarly, it cannot be valid for the final slice as in that case it must be used for toast.

The fact that the second equation begins $1 - \cdots$ suggests that a sensible approach is to begin with identifying all of the cases that prevent the r^{th} slice being used for toast.

To show the next result, first eliminate the bracketed expressions from the two equations to get an equation relating s_r and t_r. Substituting this into the second equation then gives the required result.

The formula for s_r can then easily be proven by induction and the relationship between s_r and t_r used to find the corresponding expression for t_r.

Finding expressions for the final terms of the sequences can now be achieved by finding the appropriate equations to relate the use of the final slice of bread to the previous one and substituting the expressions for s_{n-1} and t_{n-1} into these.

STEP II 2017
Hints and Solutions

Question 1

The opening "Note" clearly flags a result which will prove important in this question; this is a 'standard' result, but one that can slip by unnoticed in single-maths A-levels; it is, therefore, worth learning as an "extra", if necessary.
In (i), it should be obvious that the process of *integration by parts* is to be used and that the initial hint indicates that the "first part" must be the "arctan x" term, despite appearing as the second term of the product to be integrated. This will lead directly to the given result, following which the substitution of $n = 0$ leads to the integral $\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x$. This can be done by "recognition" (or *reverse-Chain Rule* integration) or by substitution. In the first instance, one would note that $\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,\mathrm{d}x$, where the numerator is precisely the derivative of the denominator – the standard "log. integral" form; in the second instance, the substitution $u = 1 + x^2$ will also work, though it might take just a few lines more working.
In (ii), one needs only to use the result given in (i), this time replacing n by $(n+2)$ to find an expression for $(n+3)I_{n+2}$; and then adding to it the given expression for $(n+1)I_n$. This leads to a result in which two integrals must be added to get, when simplified, $\int_0^1 \frac{x^{n+1}(1+x^2)}{1+x^2}\,\mathrm{d}x$, which should need no further comment. Setting $n = 0$ and then $n = 2$ in this result then yields a numerical answer for I_4, since I_0 has already been calculated.
In (iii), no matter how demanding the process of mathematical induction appears to be, it is very formulaic in some respects and there are always some marks to be had. To begin with, there is always the "baseline" case of (usually) $n = 1$. In this case, one must set $n = 1$ in the proposed formula and check it against the value of I_4 already known. This reveals the value of A to be $\frac{1}{4}\pi - \frac{1}{2}\ln 2$. However, since it is constant, it remains fixed during all of the remaining work and one can most easily progress through the rest of the inductive proof by simply continuing to use the letter A.
A clearly stated induction hypothesis is enormously helpful (usually replacing the n by another dummy index, k say) rather than vague statements such as "assume the result is true for $n = k$" or meaningless statements such as "assume $n = k$". We thus proceed assuming $(4k+1)\,I_{4k+1} = A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r\frac{1}{r}$.
The remainder of the proof relies more on carefulness than inspiration, especially as the process relies on exactly the same techniques as were used in part (ii), using k and $(k+1)$ in turn in the given result.

Question 2

The opening part of the question is essentially identical to the process of *composition of functions*. Moreover, if subscripts are likely to prove confusing, then begin with a statement such as "Let x_n = *X*." Thus $x_{n+1} = \dfrac{aX-1}{X+b}$ and $x_{n+2} = \dfrac{a\left(\dfrac{aX-1}{X+b}\right)-1}{\left(\dfrac{aX-1}{X+b}\right)+b}$, which simply needs tidying up. This yields $x_{n+2} = \dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}$. The "**Note**" in (i) reminds the reader that the period of a periodic sequence is the length of the *smallest* cycle of repetition; thus, we require $x_{n+2} = x_n$ but **not** $x_{n+1} = x_n$. A moment's thought should reveal it to be obvious that a constant sequence should still yield part of the same algebra which gives $x_{n+2} = x_n$ and it is worth exploring this first:

if $x_{n+1} = x_n$ then $aX - 1 = X^2 + bX \Rightarrow 0 = X^2 - (a - b)X + 1$.

One might be tempted to try to solve this (using the *Quadratic formula*, for instance), but there is really nothing to be gained by so doing, since the constant sequence is of no interest to us, only the conditions that give it (which we need to have in mind later on). Proceeding to explore $x_{n+2} = x_n$ gives us, upon collecting up, $0 = (a + b)\{X^2 - (a - b)X + 1\}$. Fortunately, the factor $(a + b)$ is readily apparent, but the quadratic factor should have been anticipated also, for the reasons outlined above. Thus, $a + b = 0$ is a necessary condition for a period 2 sequence. (There is no requirement to explore the issue of sufficiency, which could be done by setting $b = -a$ in the initial expression for x_{n+1} and then following it through to see what happens.)

Finally, we are asked to see what happens when $x_{n+4} = x_n$, and this can be done the long way round by finding $x_{n+3} = \dfrac{(a^3-2a-b)X-(a^2+ab+b^2-1)}{(a^2+ab+b^2-1)X-(a+2b+b^3)}$ and then $x_{n+4} = \dfrac{ax_{n+3}-1}{x_{n+3}+b}$, but there is at least one very obvious shortcut to what is starting to look like some complicated algebra: namely, considering the "two-step" result already found and repeating that, going from x_n to x_{n+4} via x_{n+2}. It also helps if one realises that whatever algebraic expression appears, we know that it must have the previously found factors of $(a + b)$ and $\{X^2 - (a - b)X + 1\}$ within it.

Question 3

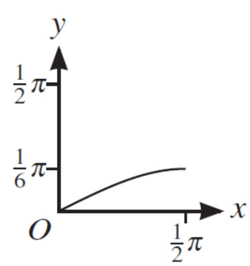
Whilst this question might appear somewhat daunting on first reading, it involves little more than an understanding of the sine curve and the key results that relate to "angles in all quadrants".

To begin with, "$\sin y = \sin x \Rightarrow y = x$" is the kind of response expected from those candidates who have failed to understand that there are many functions (even the simple ones covered at A-level) that don't map "one-to-one". Though the general formula "$\sin y = \sin x \Rightarrow y = n\pi + (-1)^n x$" might be unfamiliar, it only says that if $\sin y = \sin x$ (imagining x to be positive and acute) then y could be any "first quadrant" equivalent of x … an even multiple of π plus x … or a "second quadrant" supplement of x … an odd multiple of π minus x. (Apart from that, y's corresponding to non-acute x's arise from application of the same principles, with "quadrants" taking care of themselves due to the symmetries of the sine curve.)

In (i), it is then found that the general formula (or its equivalent in two bits) gives three graphs that are of interest here: $n = -1$ ($y = -\pi - x$), $n = 0$ ($y = x$) and $n = 1$ ($y = \pi - x$), all of which give straight-line segments in the interval required.

In (ii), there is rather less thinking to be done – at least to begin with – since one is only required to differentiate twice and do some tidying up. This calculus can be done implicitly or directly after rearranging into an arcsine form. That is not to say that it is easy calculus, since the *Product*, *Quotient* and *Chain* rules can all play a part in the processes that follow.

In sketching the graph, one should start simple and work up. Initially, $\frac{dy}{dx} = \frac{1}{2}$ at (0, 0), with the curve increasing to a maximum at ($\frac{\pi}{2}$, $\frac{\pi}{6}$), since $\frac{d^2y}{dx^2} < 0$. This gives



Thereafter, for whatever "other" bits there are to the curve, these follow from symmetries, applied to the above portion: namely, reflection symmetry in $x = \frac{\pi}{2}$; rotational symmetry about *O*; and reflection symmetry in $y = \pm\frac{\pi}{2}$.

Part (iii)'s graph follows by applying the result $\cos y \equiv \sin(\frac{\pi}{2} - y)$.

Question 4

This is an interesting question and very straightforward in some respects. To begin with, you are told in part (i) that f(x) = 1. At this point, it would be wise to write down what the *Schwarz inequality* gives in this case:

$$\left(\int_a^b \mathrm{g}(x)\,\mathrm{d}x\right)^2 \le \left(\int_a^b 1\,\mathrm{d}x\right)\left(\int_a^b [\mathrm{g}(x)]^2\,\mathrm{d}x\right);\ \text{i.e.}\ \left(\int_a^b \mathrm{g}(x)\,\mathrm{d}x\right)^2 \le (b-a)\left(\int_a^b [\mathrm{g}(x)]^2\,\mathrm{d}x\right).$$

A few moments of careful thought (inspecting the given answer) should make you realise that $a = 0$, $b = t$ give the terms $(b - a) = (t - 0)$ *and* $(\mathrm{e}^t - \mathrm{e}^0)$ when g(x) = e^x. Following it through from there is relatively routine, provided one spots the *difference-of-two-squares factorisation* and that we can divide throughout by $(\mathrm{e}^t + 1)$, which is guaranteed to be positive (an important consideration when dealing with inequalities).

In (ii), it is (again) best to start by seeing how things appear when you have used the given information that f(x) = x and, by clear implication, $a = 0$ and $b = 1$:

$$\left(\int_0^1 x\,\mathrm{g}(x)\,\mathrm{d}x\right)^2 \le \left(\int_0^1 x^2\,\mathrm{d}x\right)\left(\int_0^1 [\mathrm{g}(x)]^2\,\mathrm{d}x\right);\ \text{i.e.}\ \left(\int_0^1 x\,\mathrm{g}(x)\,\mathrm{d}x\right)^2 \le \tfrac{1}{3}\left(\int_0^1 [\mathrm{g}(x)]^2\,\mathrm{d}x\right).$$

The $\mathrm{e}^{-\frac{1}{4}}$ in the given answer, along with the fact that $\left(\mathrm{e}^{-\frac{1}{4}x^2}\right)^2 = \mathrm{e}^{-\frac{1}{2}x^2}$ should point the way towards choosing g(x) = $\mathrm{e}^{-\frac{1}{4}x^2}$. Following this through carefully again yields the required result.

The result in part (iii) clearly requires the use of the Schwarz inequality twice, once each for the right- and left-halves of the given result. Setting $\mathrm{f}(x) = 1$, g(x) = $\sqrt{\sin x}$, $a = 0$ and $b = \frac{1}{2}\pi$ leads to $\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,\mathrm{d}x \le \sqrt{\frac{\pi}{2}}$. However, the left-hand half of the result does require a bit more thought and, preferably, familiarity with the integration of trig. functions where powers of $\sin x$ (in this case) appear along with its derivative, $\cos x$. The real clue is that, for the $\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,\mathrm{d}x$ to appear on the other side of the inequality to the one found using the obvious candidates that led to the right-hand half of the result, $\sqrt{\sin x}$ must now be the result of the squaring process. It is then experience (or insight) that suggests setting f(x) = $\cos x$, g(x) = $\sqrt[4]{\sin x}$, $a = 0$ and b = $\frac{1}{2}\pi$. You will then find that the LHS of the *Schwarz inequality* requires the integration of a function of sin x multiplied by its derivative, cos x; and this (as in Q1) can be done by "recognition" or substitution. (You will also need to be able to integrate $\cos^2 x$, which calls upon the use of the standard *double-angle formula* for cosine.)

Question 5

In many ways, much of this question is also relatively routine. Find $\frac{dy}{dx}$ for the gradient of the tangent; find its negative reciprocal for the gradient of the normal and then any one of a number of formulae for the equation of a line. At some stage, you will need to replace t by p for the normal at P and then replace x and y in this equation by an^2 and $2an$ for another point on the curve. Solving for n in terms of p – noting that the factor $(n - p)$ must be involved somewhere, since $n = p$ must be one solution to whatever equation arises as the line is already known to meet the curve at P – should then yield the given answer.

In (ii), employing the distance formula $PN^2 = (x_P - x_N)^2 + (y_P - y_N)^2$ is clearly the way forwards, as is replacing n by $-\left(p + \frac{2}{p}\right)$ at some stage of the proceedings. The rest is just careful algebra. Differentiating the given expression for PN^2 with respect to p is routine enough, in principle, and it is then only required to justify that the (only) values of p that arise will give minimum points. One could use the *first-derivative test* (looking for a change of sign), the *second-derivative test* (examining its sign) or argue from the shape of the curve $(y =)\ \frac{16a^2}{p^4}(p^2 + 1)^3$ … which is symmetric in the y-axis, asymptotic to the y-axis for small values of p, and can be arbitrarily large as $|p| \to \infty$; thus, any turning-points must be minima.

For part (iii), one starts by noting that PQ and NQ are perpendicular (since $\angle PQN = 90°$, by "Angle in a semi-circle"). Then, setting the products of their gradients, $\frac{2}{p+q}$ and $\frac{2}{n+q}$, equal to – 1, replacing n by $-\left(p + \frac{2}{p}\right)$ once again, and using $p^2 = 2$, takes you almost the whole way there: $q^2 = \frac{2q}{p}$. From this point, we have $q = 0$ or $q = \frac{2}{p} = \pm\sqrt{2}$. Finally, these final two cases should be eliminated by noting that they give $q = p$, i.e. $Q = P$, which is not the case as they are being taken to be distinct points.

Question 6

This question is about two different types of proof – induction and direct manipulation. Both of which in isolation are generally well understood, but it is very easy to get lost in the algebra, especially with the added complication of inequalities.
Part (i) explicitly required induction, so there is a standard procedure to follow – check it works when $n = 1$, assume it works when $n = k$ and show that this leads to it being true when $n = k + 1$. Only the last part causes any issue. There is some fairly subtle logic: using the $n = k$ assumption it can be shown that

$$S_{k+1} \leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$$

However, what is really needed is:

$$S_{k+1} \leq 2\sqrt{k+1} - 1$$

One way in which this can be established (technically a sufficient, but not necessary condition) is if it can be shown that

$$2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$$

A bit of tidying and rearranging shows that this is equivalent to

$$4k^2 + 4k \leq 4k^2 + 4k + 1$$

which is "obviously" true. You might worry a little about the fact that the equality is never satisfied, but showing that the strict inequality is true is sufficient.

The first part of part (ii) is also just about squaring up and showing that the statement is equivalent to an "obviously" true statement (in this case that $16k^3 + 24k^2 + 9k + 1 > 16k^3 + 24k^2 + 9k$). No induction required! But why are we being asked to do this....?

In the final part we first need to come up with a conjecture. A reasonable place to start is to try $n = 1$, so that

$$1 \geq 2.5 - C$$

For this to work we need that $C \geq 1.5$ so $C = 1.5$ is the smallest value that works for S_1. However will this work for all subsequent S_n too? It turns out that it does, but that requires proving the conjecture

$$S_n \geq 2\sqrt{n} + \frac{1}{\sqrt{n}} - 1.5$$

This requires induction, following a very similar argument to part (i). One line of the algebra in the proof requires $(4k + 1)\sqrt{k+1} \geq (4k + 3)\sqrt{k}$, which can be done using the initially unimportant fact at the beginning of part (ii) – always look out for making links between the different parts of questions!

Question 7

In this question the difficulty is being able to see that some result is "obviously" true but then having great difficulty in justifying it from particular starting-points: it is not enough to make a true statement (especially when it is given in the question) … one must justify it fully from given, or known, facts and careful deductive reasoning.

Here, in (i), it is known that, for $0 < x < 1$, x is positive and $\ln x$ is negative. Thus $0 > x \ln x > \ln x$ can be deduced by multiplying the first inequality throughout by a negative quantity (remembering that this reverses the direction of inequality signs). This is just $\ln 1 > \ln x^x > \ln x$ and, since the logarithmic function is strictly increasing, $(1 >)\ \mathrm{f}(x) > x$. A more complete argument along similar lines shows that $x < \mathrm{g}(x) < \mathrm{f}(x)$. The final part requires no further justification; since for $x > 1$, $\ln x > 0$ we now have $x < \mathrm{f}(x) < \mathrm{g}(x)$.

For part (ii), it is customary to use logs first and then differentiate implicitly.

In (iii), only an informal understanding regarding the justification of limits is expected, but one still should have a grasp as to how things should be set out. Here, something along the lines of

$$\mathcal{L}im_{x \to 0} \left(\mathrm{f}(x)\right) = \mathcal{L}im_{x \to 0} \left(\mathrm{e}^{x \ln x}\right) = \mathcal{L}im_{x \to 0} \left(\mathrm{e}^{0}\right) = 1$$

and so $\mathcal{L}im_{x \to 0} \left(\mathrm{g}(x)\right) = \mathcal{L}im_{x \to 0} \left(x^{\mathrm{f}(x)}\right) = \mathcal{L}im_{x \to 0} \left(x^{1}\right) = 0$

would be expected.

In (iv), the use of calculus is the most straightforward approach, differentiating $y = \frac{1}{x} + \ln x$ for $x > 0$ and showing that it has a unique minimum turning point at $(1, 1)$. This is then fed in to the derivative of $\mathrm{g}(x)$ – again using the logarithmic form and implicit differentiation – along with a simple observation that squares are necessarily non-negative and this all falls nicely into place. Most of what is required in order to sketch x, $\mathrm{f}(x)$ and $\mathrm{g}(x)$ has already been established and all that is left is to put it together in a sensibly-sized diagram.

Question 8

Although vectors expressed in general terms are not handled well by the majority of STEP candidates, such questions invariably involve little that is of any great difficulty. If one is sufficiently confident in handling vectors, this question is perhaps the easiest on the paper. The only things involved in this question are the equations of lines in the standard vector form $\mathbf{r} = \mathbf{p} + t\,\mathbf{q}$ and the use of the scalar product for finding angles (in particular the result that, for non-zero vectors $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{p} \bullet \mathbf{q} = 0 \Leftrightarrow \mathbf{p}$ and $\mathbf{q}$ are perpendicular).
Thus, we have the line through A perpendicular to BC is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ and the line through B perpendicular to CA is $\mathbf{r} = \mathbf{b} + \mu\,\mathbf{v}$, which meet when

$$\mathbf{a} + \lambda\,\mathbf{u} = \mathbf{b} + \mu\,\mathbf{v} \Rightarrow \mathbf{v} = \frac{1}{\mu}(\mathbf{a} - \mathbf{b} + \lambda\mathbf{u}).$$

Since $\mathbf{v}$ is perpendicular to CA, $(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u}) \bullet (\mathbf{a} - \mathbf{c}) = 0$ which leads to a scalar expression for λ and hence a vector expression for $\mathbf{p} = \mathbf{a} + \lambda\,\mathbf{u}$.
Next, $\overrightarrow{CP} = \mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} + \lambda\mathbf{u}$, and
$\overrightarrow{CP} \bullet \overrightarrow{AB} = (\mathbf{a} - \mathbf{c} + \lambda\mathbf{u}) \bullet (\mathbf{b} - \mathbf{a}) = (\mathbf{a} - \mathbf{c}) \bullet (\mathbf{b} - \mathbf{a}) + \lambda\mathbf{u} \bullet (\mathbf{b} - \mathbf{a})$.
Now $\mathbf{u} \bullet (\mathbf{b} - \mathbf{c}) = 0$ since $\mathbf{u}$ is perpendicular to $BC \Rightarrow \mathbf{u} \bullet \mathbf{b} = \mathbf{u} \bullet \mathbf{c}$. Substituting this into $\overrightarrow{CP} \bullet \overrightarrow{AB}$ leads very quickly to the required zero for the perpendicularity result required.
$(\mathbf{a} - \mathbf{c}) \bullet (\mathbf{b} - \mathbf{a}) + \lambda\mathbf{u} \bullet (\mathbf{c} - \mathbf{a})$.

[**Note:** those readers familiar with the common geometric "centres" of triangles will, no doubt, have spotted that this question is about nothing more than the *orthocentre* of a triangle; that is, the point at which the three altitudes meet. In this question, you are given two altitudes; find their point of intersection, and then show that the line from the third vertex through this point meets the opposite side at right-angles.]

Question 9

Before doing anything else in a question like this you need to draw a BIG diagram with all relevant forces labelled.
With nearly all questions like this it is a matter of setting up equations by resolving forces and taking moments. The main tactical decision is about in which direction to resolve forces and about which point to take moments. Because the equation we are trying to find in part (i) does not involve the weight or the normal reaction force of the cylinder and the ground there is a strong indication that we will be resolving horizontally. Focussing on just one cylinder we get:

$$F + F_1 \cos\theta = R \sin\theta \qquad (1)$$

where F_1 is the friction between the plank and the cylinder.
We also want to find a point that is in line with these forces so that when we take moments about that point the forces will not feature. A natural point to choose is the centre of the cylinder. Taking moments about this point gives:

$$F.r = F_1.r$$

So $F = F_1$. Substituting this into equation (1) gives the required result.
There are several ways in which an inequality can arise in mechanics, but in questions on friction a good possibility is using the fact that $F_1 \leq \mu R$. Combining this with the above results gets the required inequality.

Part (ii) does bring in the normal reaction with the floor, so resolving vertically will be a useful tool. For the cylinder this gives:

$$W \ = \ N - \ R \cos\theta - \ F_1 \sin\theta$$

where W is the weight of the cylinder. Unfortunately, we do not want W in our expression, but we do need to bring in a k. The easiest way to do this is to resolve vertically for the plank:

$$kW \ = \ 2R \cos\theta \ + \ 2F \sin\theta$$

Combining these two expressions with the result from part (i) and the fact that $\cos^2\theta + \sin^2\theta = 1$ leads to the required result.
If there is no slipping then $F \leq \mu N$. Substituting in the result we have just obtained turns this into something that can be rearranged into something similar to the last result in part (i):

$$2k \sin\theta \ \leq (k \ + \ 2)(1 + \cos\theta) \qquad (2)$$

It is always important when dealing with these algebraic expressions to try to make links. We can use the final result in part (i) to show that (by multiplying by k):

$$2k\sin\theta \le k(1+\cos\theta)$$

But $k(1+\cos\theta)$ must be less than $(k+2)(1+\cos\theta)$, therefore as long as the inequality from (i) is true then inequality (2) will be satisfied.
Sometimes it is important to see the thrust of a question. The first two parts have been steering you towards the idea that the important inequality is $2\sin\theta \le 1+\cos\theta$. Our task is to now turn this into an inequality involving only $\sin\theta$. It is tempting to subtract 1 from both sides and square, as that will lead to an inequality involving only sines. However that is technically flawed: we cannot easily square expressions which are sometimes positive and sometimes negative [i.e., it is not generally true that $a < b$ leads to $a^2 < b^2$; for instance, consider $-2 < 1$]. It is better to square up the original expression as it is, in the context of the question, never negative. After a bit of manipulation, it becomes:

$$0 \le (5\cos\theta - 3)(\cos\theta + 1)$$

From this it can be deduced that $\cos\theta \ge \frac{3}{5}$, and in turn a graphical argument leads to the required result.
This then needs to be related to the physical situation. By finding an appropriate right-angled triangle you can show that $\sin\theta = \frac{r-a}{r}$, which leads to the required result.

Question 10

With questions involving lots of "show that" work it is particularly important to not simply write down expressions which are true, but could have come from "working backwards".

The first critical idea is that the work done by the car is $\int_0^d F \,\mathrm{d}x$ where F is the force exerted by the car.

The second critical idea is to use Newton's 2nd Law:

$$ma = F - (Av^2 + R)$$

To obtain the second integral a change of variable is required. This needed a clear explanation of how to change both the limits and the $\mathrm{d}x$.

In part (i) the integral had to be split up to reflect the two parts of the journey. In theory either the integral with respect to x or with respect to v could be used, but you might think that the fact that you were led towards the integral with respect to v just above suggests that it would be the better choice, and this is indeed the case. It was also important to explain why the $R > ma$ condition was needed.

In part (ii) the given condition needs to be interpreted in terms of the speed at which the force is zero. This needs to be compared to w to check that it is achieved. Then some more integrals similar to part (i) and some algebra leads to the required result.

Question 11

As with most mechanics questions, a large clear diagram is very useful. Although not mentioned in the question, defining the angle of projection is a very good idea in projectile questions.
Conserving energy provides a fairly standard start to this question. We then needed to transfer to kinematics to introduce angles. An important decision needs to be made about where to set the origin. It turns out that the top of the first wall makes a very sensible choice. Standard kinematic equations can be used to write the vertical and horizontal displacement when the particle passes over the second wall. Eliminating the time from these equations and using the result from the first part leads to a familiar looking trigonometric expression.
To obtain the distance of A from the foot of the wall it is useful to find the angle of projection. To do this it is useful to find something that doesn't change to form into an equation; in this case the horizontal component of the velocity. This leads to finding the cosine of the angle of projection. Using a trigonometric identity can turn it into the sine of the angle. You can then use a kinematics equation to describe the vertical displacement, finding a quadratic equation for the time taken to get to the height of the wall. This time can be used to find the horizontal displacement.
Part (ii) follows a similar pattern to part (i). Energy considerations can be used to find the speed over the first wall. Then kinematics equations (or more directly the trajectory equation) can used to form a quadratic equation in the tan of the angle passing over the first wall if it just passes the second wall. Examining the discriminant (after a fair amount of algebra) shows that this equation does not have a solution, so the particle cannot pass over the second wall.

Question 12

Part (i) required thinking about the different ways in which the total number of fish caught could be n – for each value of X, there is a corresponding value of Y. This leads to a sum. Each probability can be written using the formula for the Poisson distribution. It is useful to have an idea of what you are trying to get to (a $\text{Po}(\lambda + \mu)$ distribution). Pulling out some common factors leaves something very close to a binomial expansion of $(\lambda + \mu)^n$. Artificially pulling out another factor of $\frac{1}{n!}$ leaves exactly the required expansion.
Part (ii) starts by turning the situation described into a probability statement, then using the formula for conditional probability. Substituting in the expressions from the Poisson distributions and a little algebra leads to a standard binomial expression.
Part (iii) is all about linking with part (ii). When the first fish is caught the total number of fish caught is one, and you want to know the probability that Adam caught it.
Part (iv) requires some quite subtle thinking. The expected waiting time can be split into the expected waiting time with Adam first or with Eve first. Some careful thought is required to realise that, for example, the waiting time with Adam first can be broken down into the time for a fish to be caught followed by the time for Eve to catch a fish.

Question 13

The first step of part (i) is to find an expression for the probability of getting the correct key on the k^{th} attempt. This can be done from a tree diagram or by using the geometric distribution. From these probabilities an expression for the expectation can be found which is strongly related to the binomial expansion suggested.
Part (ii) also needs to start with an expression for the probability of getting the correct key on the k^{th} attempt. This can be found by telescoping expressions from a tree diagram or just using the symmetry of the situation: each possible selection is equally likely to find the correct key. An expression can again be found and simplified for the expectation.
Part (iii) : A tree diagram type approach forms a series of telescoping fractions, simplifying to the given expression. Pulling out a factor of $(n - 1)$ from the expression for the expectation leaves a series of partial fractions which can be written as the difference between the infinite sum given and a finite sum. The difference between an infinite quantity and a finite quantity must be infinite.

STEP III 2017
Hints and solutions

Question 1

The first result is simply obtained by expanding the bracketed expression on the right-hand side using the definition of the binomial coefficients, and then combining the fractions using the lowest common denominator. $\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}C_{r+1}}$ is determined by employing the first result of the question, finding that the terms telescope and then observing that ${}^{n+r}C_r \to \infty$ as $n \to \infty$, to give the answer $\frac{r+1}{r}$. The deduced result is obtained by letting $r = 2$ in the previous result and subtracting the first term of the sum in the general result. The first inequality of (ii) can be obtained by expanding ${}^{n+1}C_3$ as $\frac{n^3-n}{3!}$, observing that $\frac{n^3-n}{3!} < \frac{n^3}{3!}$ and rearranging. Similarly, $\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} - \frac{5!}{n^3}$ is $\frac{-480}{n^3(n^2-1)(n^2-4)}$ which is negative as $n \geq 3$ and so the denominator is positive, leading to the second inequality. The first numerical inequality in the final result is obtained from the second inequality of part (ii) using the final result of (i) for the first summed term, the penultimate result of (i) for the second summed term (adjusting the index over which it is summed), and including the terms for $n = 1$ and $n = 2$. The second numerical inequality in the final line is obtained from the first inequality of part (ii), again including the two extra terms and using the final result of (i).

Question 2

(i) is simply obtained by applying $e^{i\theta}$ to $z - a$. Using the result of (i) twice for SR and for a rotation about c and equating, both sides can be multiplied by $-e^{\frac{-i(\varphi+\theta)}{2}}$. In the case $\varphi + \theta = 2n\pi$, $\left(1 - e^{i(\varphi+\theta)}\right) = 0$, so c cannot be found, and then SR is a translation by $(b-a)\left(1 - e^{i\varphi}\right)$. If $RS = SR$, and if $\varphi + \theta = 2n\pi$, then

$(b-a)\left(1 - e^{i\varphi}\right) = (a-b)\left(1 - e^{i\theta}\right)$ and so either $a = b$ or if $a \neq b$, $\theta = 2m\pi$. If $\varphi + \theta \neq 2n\pi$, $a = b$, $\theta = 2n\pi$, or $\varphi = 2n\pi$

Question 3

Writing down the sum of the three roots of the cubic gives an expression which must be q in the quartic and $-A$ in the cubic. In the specific case, the cubic equation is

$y^3 - 3y^2 - 40y + 84 = 0$ which has roots -6, 2 and 7, and thus $\alpha\beta + \gamma\delta = 7$. Expanding the expression whose value is required in (ii) gives q without the two terms whose sum has just been found, and hence -4. Equally the product of those two terms is s (10), and so a quadratic equation for them gives $\alpha\beta = 5$, the larger of the two roots. Using the first result of (ii) with the knowledge that $(\alpha + \beta) + (\gamma + \delta) = p = 0$, and that

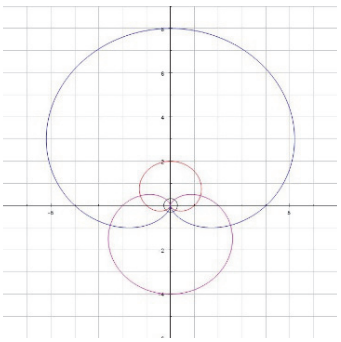
$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 6$, leads to the results that α and β are the roots of $t^2 + 2t + 5 = 0$ and γ and δ are the roots of $t^2 - 2t + 2 = 0$. Hence the four roots of the quartic are $1 \pm i$, $-1 \pm 2i$.

Question 4

Letting $\log_a \mathrm{f}(x) = z$, $\mathrm{f}(x) = a^z = \mathrm{e}^{z \ln a}$ (using the result from the formula book) and so, $\ln \mathrm{f}(x) = z \ln a = \ln a \ \log_a \mathrm{f}(x)$, which substituted in the geometric mean definition simplifies rapidly to the required result. Part (ii) can be obtained by substituting for $\mathrm{h}(x)$ in the expression for $\mathrm{H}(y)$ and then manipulating using the logarithm of a product. Part (iii) can be obtained using the result of part (i) with $a = b$, and then checking separately that the simple case $b = 1$ works. In part (iv), setting the defined expression for the geometric mean of $\mathrm{f}(x)$ equal to $\sqrt{\mathrm{f}(y)}$, and taking the logarithm of both expressions yields, after minor rearrangement, $\int_0^y \ln \mathrm{f}(x)\mathrm{d}x = \frac{y}{2} \ln \mathrm{f}(y)$. Differentiating this with respect to y, and again rearranging, leads to the first order separable differential equation $\frac{\mathrm{f}'(y)}{\mathrm{f}(y) \ln \mathrm{f}(y)} = \frac{1}{y}$, which integrates to $\ln \ln \mathrm{f}(y) = \ln y + c$ leading to the desired result by judicious choice of c.

Question 5

Converting polar coordinates to Cartesian and differentiating each of x and y with respect to θ gives a fraction which simplifies to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{f} + \mathrm{f}' \tan\theta}{-\mathrm{f}\tan\theta + \mathrm{f}'}$ once the numerator and denominator have been divided by $\cos\theta$. Setting the product of two such expressions for f and g equal to negative 1 and simplifying leads to $(\mathrm{fg} + \mathrm{f}'\mathrm{g}')\sec^2\theta = 0$ and hence the desired result. Substituting for g in the displayed result and solving the resulting first order differential equation for f (by integrating factor or separation of variables) leads to $\mathrm{f}(\theta) = \left(\frac{k\cos^2\theta}{1+\sin\theta}\right)$ which simplifies by eliminating $\cos^2\theta$ in favour of $\sin^2\theta$, and hence using the given point, $\mathrm{f}(\theta) = 2(1 - \sin\theta)$.



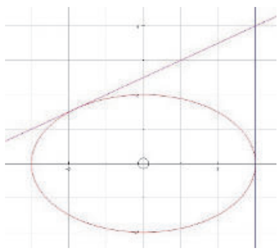
Question 6

The appropriate substitution for part (i) was $u = v^{-1}$, and having changed variable the resulting integral has limits x^{-1} and ∞ , which can be expressed as the difference of two integrals with these as their upper limits and zero as their lower. To obtain $\frac{\mathrm{d}v}{\mathrm{d}u}$ in part (ii), one method is to make a (the constant) the subject of the formula and to differentiate with respect to u; an alternative is to differentiate v directly, to multiply numerator and denominator by $(1 + u^2)$, to expand the numerator and then to express it as $(1 - au)^2 + (u + a)^2$ leading to the desired result. Applying this substitution to the defined $\mathrm{T}(x)$, results in an integral which can be expressed again as the difference of two integrals as in part (i). Taking the result of (i) and rearranging to make $\mathrm{T}(x^{-1})$ the subject, $\mathrm{T}(x)$ can be substituted for using the result just found and in turn $\mathrm{T}(a)$ can be replaced using the result of (i) with x as a. The final result of (ii) is achieved by letting $y = x^{-1}$, and $b = a^{-1}$ in that just found. Throughout, it is important that the conditions expressed as inequalities are substantiated. To find $\mathrm{T}(\sqrt{3})$ in (iii), apply the final result of (ii) letting $y = b = \sqrt{3}$, whereas to find $\mathrm{T}(\sqrt{2} - 1)$, let $x = \sqrt{2} - 1$ and $a = 1$ in the result

$\mathrm{T}(x) = \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a)$, deal with $\mathrm{T}(\sqrt{2} + 1)$ by letting $x = \sqrt{2} + 1$ in the result of part (i) and then $\mathrm{T}(1)$ using the same result but with $x = 1$.

Question 7

Showing T lies on the ellipse is merely a matter of substituting T's coordinates into the left-hand side of the ellipse equation and simplifying to equal 1. The first result of (i) is tantamount to finding the equation of the tangent at T. Parametric differentiation leads to $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{b(1-t^2)}{2at}$ giving L as $y - \frac{2bt}{(1+t^2)} = -\frac{b(1-t^2)}{2at}\left(x - \frac{a(1-t^2)}{(1+t^2)}\right)$ which (X, Y) satisfies, thus simplifying to the required result. The deduction can be made by requiring that the discriminant of the quadratic in t is positive and geometrically (X, Y) is a point outside the ellipse. Relaxing the restriction on the value of X, $X = \pm a$, so the inequality implies $Y \neq 0$ and thus there is a vertical tangent and another with one possible configuration as shown.



The first result of (ii) is obtained by considering p and q to be the roots of the quadratic in t, and hence being able to write down their product. Similarly, $p + q = \frac{2aY}{(a+X)b}$. The final result is obtained by finding expressions for y_1 and y_2 in terms of p and q respectively (without loss of generality), imposing the condition on y_1 and y_2 to get an equation in p and q, and then using the first two results from this part of the question to substitute for the product and sum of p and q.

Question 8

The stem can be achieved by adding the two summations, expanding the brackets, and observing that the resulting two summed terms telescope. (i) is simply a case of using the given expression for b_m , and letting $a_m = 1$ (or any constant) in the stem result, simplifying the left-hand side using the given note, and dividing through by $\sin\frac{1}{2}x$. Part (ii) can be obtained by letting $a_m = m$ and $b_m = \sin(m-1)x - \sin mx$ (or similarly $b_m = \cos\left(m - \frac{1}{2}\right)x$) in the stem which simplifies to give $p = -\frac{1}{4}n$ and $q = \frac{1}{4}(n+1)$.

Question 9

The first result can be obtained by writing the equation of motion for each particle separately and then adding the two equations to eliminate the unknown tension; two integrations with respect to time complete the working once the constants of integration (both zero) are evaluated. Using the result obtained, at the time T (given) with the value of x as a, y works out to also be a. As a consequence, conservation of energy has no elastic energy term at that instant, merely kinetic energy for each particle and the lost potential energy of A. Combining this equation with that obtained after the first integration in the initial result of the question gives simultaneous equations for the two speeds at that instant, and substituting for the speed of A gives a quadratic with the desired result as its repeated root.

Question 10

The first result is obtained by conserving energy for the rod and particle together (rotational kinetic energy and gravitational potential energy) and simplifying the algebra. Differentiating that result with respect to time and then simplifying gives $2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$. Alternatively, the same result can be obtained by taking moments about an axis through P. Resolving perpendicular to the rod for the particle and rearranging the equation generated yields an expression for the normal reaction, $mg\cos\theta\left(\frac{3a(2a-l)}{2(3a^2+l^2)}\right)$, having used the previously obtained expression for $\ddot{\theta}$. This is demonstrably positive under the given conditions. Resolving along the rod towards P (i.e. radially inwards) yields an expression for the friction which is simplified using the first obtained result of the question, and then applying the conditions for limiting friction yields the given result. In the case $l > 2a$, the particle loses contact immediately as the rod falls away quicker than the particle accelerates downwards; this can be shown either by considering the equation of rotational motion for the rod alone about P and finding $l\ddot{\theta} = \frac{lg}{2a}$, or by observing from previous working that the normal reaction of the rod on the particle would need to be negative for the particle to stay in contact with the rod.

Question 11

Conserving linear momentum in part (i) leads to $u = \frac{nmv}{M}$, and using this result leads directly to the displayed kinetic energy result. Conserving momentum before and after the r^{th} gun is fired gives $\big(M + \big(n - (r-1)\big)m\big)u_{r-1} = (M + (n-r)m)u_r - m(v - u_{r-1})$ which leads to the required result, and summing that result for $r = 1$ to n gives, on the right-hand side, a sum of n terms, each of which can be shown to be less than (or equal to in one case) $\frac{mv}{M}$, and hence the result. For (iii), considering the energy of the truck and the $\big(n - (r-1)\big)$ projectiles before and after the r^{th} projectile is fired,

$$K_r - K_{r-1} = \frac{1}{2}(M + (n-r)m)u_r{}^2 + \frac{1}{2}m(v - u_{r-1})^2 - \frac{1}{2}\big(M + \big(n - (r-1)\big)m\big)u_{r-1}{}^2 .$$

Simplifying this by collecting the terms $\frac{1}{2}(M + (n-r)m)(u_r{}^2 - u_{r-1}{}^2)$ leads to the printed result via the difference of two squares factorisation and use of the result from (ii). Once again, summing as before with telescoping terms leads to the second printed result, and the final inequality follows using the inequality from (ii) via a little algebraic simplification; the final step is quite a slack inequality.

Question 12

To obtain the first result of (i), sum in turn over x and y from 1 to n to obtain the total probability (1!) yields $k = \frac{1}{n^2(n+1)}$, and then sum over y. For independence, it would be necessary that $\mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x)\mathrm{P}(Y = y)$ and it is possible to simplify the algebra of this equation having substituted for each probability to see that there are numerous examples for which this does not hold. Proceeding as at the start of the question to obtain $\mathrm{E}(XY) = \frac{(n+1)(2n+1)}{6}$ and summing over x using the printed result of (i) to find $\mathrm{E}(X) = \frac{(7n+5)}{12}$ (and hence $\mathrm{E}(Y)$ also) yields, following simplification, $Cov(X, Y) = \frac{-(n-1)^2}{144}$, establishing the required result.

Question 13

To find $\mathrm{V}(x) = \sigma^2 + (x-\mu)^2$, expanding the definition of $\mathrm{V}(x)$ and then expressing $\mathrm{E}(X^2)$ in terms of variance yields the result. The result for $\mathrm{E}(Y)$ that is given in the question follows directly from the first result, and that $\mathrm{V}(x) = \frac{1}{12} + \left(x - \frac{1}{2}\right)^2$ in the uniform case, follows from previous working, applying standard knowledge of the mean and variance of the uniform distribution. In order to find the probability density function of Y, it is simplest to find the cumulative distribution function of Y first which is done by algebraic rearrangement of $\mathrm{P}\left(\frac{1}{12} + \left(X - \frac{1}{2}\right)^2 < y\right)$ and then differentiation to give $\mathrm{f}(y) = \left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$, $\frac{1}{12} \leq y \leq \frac{1}{3}$ and 0 otherwise. The final verification is conducted by integration of $y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$ with suitable limits, which can be done by numerous methods such as expressing the function as $\left(y - \frac{1}{12}\right)\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} + \frac{1}{12}\left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$ or change of variable, of which, letting $u^2 = y - \frac{1}{12}$ is just one example.

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