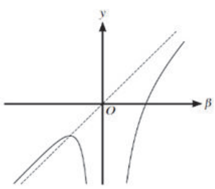
STEP MATHEMATICS 3

2018

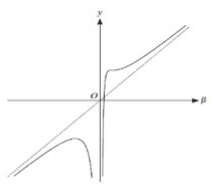
Hints and Solutions

Question 1

Differentiating $f(\beta)$ and setting equal to zero yields a cubic equation with a single real root, the quadratic factor being demonstrated to be non-zero either by completing the square or considering the discriminant. The stationary point is thus $(-1,-1)$ and with the asymptote $y=\beta$ the sketch results.



The same strategy for $g(\beta)$ yields a cubic equation with three real roots, two being coincident, and thus a maximum $\left(-2,-\frac{15}{4}\right)$ and a point of inflection $(1,3)$.



Employing Vieta's formulae gives $u+v+\frac{1}{uv}=-\alpha+\frac{1}{\beta}$ and $\frac{1}{u}+\frac{1}{v}+uv=\frac{-\alpha}{\beta}+\beta$. The first condition of part (iii) enables an expression for α to be obtained in terms of β, and so the subject of the required inequality is $f(\beta)$. When the discriminant condition is employed to impose the reality of u and v , the resulting cubic inequality has a quadratic factor which should be demonstrated to be positive (similarly to part (i)) and hence $\beta \leq 1$ which by reference to part (i), gives the requested result. Part (iv) follows the same strategy as (iii) except that it makes use of $g(\beta)$ and the reality condition cubic inequality has a squared linear factor which allows $\beta=1$ as well as $\beta \leq \frac{1}{4}$, and so by reference to the sketch, the greatest value is 3. The reader might like to consider what effect it would have on parts (iii) and (iv) if the extra condition $u \neq v$ were to be imposed.

Question 2

(i) is obtained by differentiating the defined function y_n using the product rule and a little tidying up of z and its differential. The inductive step in the proof of part (ii) can be established by differentiating the assumed case and then removing the three differentials using the result of part (i); the base case is established by obtaining $y_1 = 2x$ and $y_2 = -2 + 4x^2$ from the original definition. The deduction in (ii) is most simply obtained by eliminating x between the result for y_{n+1} just obtained and the similar one for y_{n+2}. Part (iii) is obtained by using the deduction of part (ii) to establish the desired induction, the base case being obtained using the same results used for the base case in part (ii)'s induction.

Question 3

Differentiating and equating coefficients generates three simultaneous equations, two of which are in b and c only, and can be solved by substituting for one of these variables, then giving a from the third; $a = 1 + p \mp q$, $= 1 \pm 2q$, $c = \mp q - p$. Part (i) makes use of the form obtained in the stem which can be then integrated twice, with a minor algebraic rearrangement after each integration, with different cases arising as $b = 1$ or not; the solutions are

$y = Ax^{p+q} + Bx^{p-q}$ if $q \neq 0$, and $y = Ax^p \ln x + Bx^p$ if $q = 0$. Part (ii) proceeds similarly, except that a, b, and c are simplified as $q = 0$. However, two cases arise again and so

$y = \frac{x^n}{(n-p)^2} + A\, x^p \ln x + Bx^p$ if $\neq p$, and $y = x^n \frac{(\ln x)^2}{2} + Ax^n \ln x + Bx^n$ if $n = p$.

Question 4

The equation of the tangent to the hyperbola at P is found in the usual way, having obtained the gradient through differentiation. In part (i) the points S and T can be found by solving simultaneously the equations for each of the given lines with that for the tangent and their midpoint is found to be P, using a Pythagorean trigonometry result to simplify the common denominators. Solving simultaneous equations using the equations of the two tangents gives $x^2 = \left[a\frac{(\sin\varphi\cos\theta - \sin\theta\cos\varphi)}{(\sin\varphi - \sin\theta)}\right]^2$ and $y^2 = -a^2\sin\theta\ \sin\varphi\left[\frac{(\cos\theta - \cos\varphi)}{(\sin\varphi - \sin\theta)}\right]^2$, having eliminated b from the latter expression using the condition that the tangents are perpendicular; that same condition and the knowledge that $a > b$ not only justifies that such tangents exist, but also that the denominator of the two results found is non-zero. Adding the two results, expanding brackets in the numerator, and then removing any cos squared terms in favour of $\sin$ squared terms leaves an expression which cancels with the denominator, and the resulting simple expression can then be seen to yield the desired result.

Whilst not within the remit of the question posed, an elegant method of obtaining the result of part (ii) is to consider the solution of the hyperbola equation with the equation of a general straight line through a point on the hyperbola and another point. Solving simultaneously for (say) the x-coordinate of the meeting of the curve and line and imposing that the solutions to the quadratic must be coincident for a tangent yields a quadratic equation for the gradients, and as the tangents are perpendicular, the product of those gradients is -1 giving the desired result without recourse to trigonometry.

Question 5

As with many inequalities, rather than proving that one expression is greater than or equal to another, it is easier to subtract and produce a single expression that one requires to show is greater than or equal to zero. Substituting for the As, simplifying and dividing by $\mathrm{G_k}$ $(\mathrm{G_k} > 0)$ gets most of the way to obtaining the first result, having demonstrated that $\frac{\mathrm{G_{k+1}}}{\mathrm{G_k}} = \lambda_\mathrm{k}$ en route, and observing the reversibility of the argument throughout. Part (ii) can be simply shown using differentiation to find and justify that there is a single stationary point for $\mathrm{f(x)}$, that it is a minimum (zero) and that it occurs for $\mathrm{f(x)}$. Part (iii) (a) can be deduced using part (i), observing that the condition for (i) is met by use of part (ii). There are several different but equivalent arguments that can be used for part (iii) (b). Assuming $\mathrm{A_k} = \mathrm{G_k}$ for some k , then by part (a), and (i), it can be shown that $\mathrm{A_{k-1}} = \mathrm{G_{k-1}}$, but also that $\lambda_{\mathrm{k-1}} = 1$ and so $\mathrm{a_k} = \mathrm{G_{k-1}}$. Thus if $\mathrm{A_n} = \mathrm{G_n}$, the argument just employed obtains $\mathrm{A_k} = \mathrm{G_k}$ and $\mathrm{a_k} = \mathrm{G_{k-1}}$ for all k , for $\mathrm{k} = 1$ to n . The final step of the argument follows simply.

Question 6

The very first result is very simple bookwork, and easily justified in numerous ways. Equating the expression to its conjugate invoking its reality, employing the properties of the conjugates of sums, products etc. and algebraic rearrangement yields the second result of part (i). Substitution for the conjugates of a and c using the unit circle property gives the final result of (i) after some algebraic rearrangement including division by $(\mathrm{c} - \mathrm{a})$ which can be justified as non-zero. Use of Q lying on AC and part (i)'s result and then similarly Q lying on BD gives two equations which can be combined linearly to give the required equation for q^* in (ii). Combining the two equations linearly can also give an equation for q , which can be added and rearranged to that already found to give the desired second result of (ii). The first result of (iii) employs the result from (i) with P lying on AB and bearing in mind the reality of p. Repeating this for P lying on CD gives a similar result. Multiplying the final result of (ii) by p gives two expressions which can be simplified by the two just written down, and a little simplifying algebra allied to the legitimate division by $(\mathrm{ac} - \mathrm{bd})$ finishes the question. As some candidates recognised, but then did not score particularly well as they quoted results without justifying them, this question is about the topic of pole and polar.

Question 7

Expressing $\cot\theta$ as $\frac{\cos\theta}{\sin\theta}$ and employing De Moivre's theorem gives the first result. Expanding the left-hand side of the first equation binomially and collecting like terms for the odd powered terms and dividing by 2i then simplifies to the left-hand side of the second result, which then equals zero if $(2n+1)\theta = m\pi$ where $m = 1, 2, \ldots, n$ and satisfies the initial condition. Part (ii) is obtained by considering the sum of roots of the equation just obtained in (i). The first result of (iii) can be derived from the given small angle inequality taking a little care with positivity when reciprocating and squaring, and then using the appropriate Pythagorean trigonometrical identity. Summing the inequalities for $m = 1$ to n, rearranging to give the required object of the summation and using the value of θ as defined for part (i), gives bounds of $\frac{n(2n-1)}{3(2n+1)^2} \times \pi^2$ and $\frac{n(2n+2)}{3(2n+1)^2} \times \pi^2$, both of which tend to the desired limit as $n \to \infty$.

Question 8

The first result in part (i) can be achieved by interpreting the sum of integrals as a single integral with limits of 1 and ∞ , and by making the change of variable, $\mathrm{y} = \mathrm{x}^{-1}$. The deduction can be made by splitting the integral into partial fractions, changing the variable using $\mathrm{y} = \mathrm{x} + 1$ in the integral of the first fraction, employing the periodicity condition and then telescoping the sums to leave the single required integral. The first integral of (ii) uses the result just obtained in (i) followed by observing that $\{\mathrm{x}\} = \mathrm{x}$ for the range of the integral, and then integrating normally to obtain the answer $1 - \ln 2$. For the second integral, it needs to be observed that

$\{2(\mathrm{x} + 1)\} = \{2\mathrm{x} + 2\} = \{2\mathrm{x}\}$ and the integral making use of the result for part (i) can be split into two integrals with limits 0 to $\frac{1}{2}$, and $\frac{1}{2}$ to 1 to deal with the two different explicit formulae for $\{2\mathrm{x}\}$; the result can be simplified to $2 + \ln\frac{3}{16}$ or a couple of equally simple equivalents.

Question 9

Newton's Experimental Law of Impact gives $\mathrm{v_{n-1} - u_{n-1} = e(v_{n-2} + u_{n-2})}$ and conserving momentum (once a factor of m has been cancelled)

$\mathrm{kv_{n-1} + u_{n-1} = u_{n-2} - kv_{n-2}}$ for the $(n-1)^{\text{th}}$ collision between P and Q; $\mathrm{v_{n-2}}$ can be eliminated between these two equations. Similarly, $\mathrm{v_n}$ can be eliminated between the corresponding equations for the n^{th} such collision. Then, the two equations that have been obtained can be manipulated to eliminate $\mathrm{v_{n-1}}$ and produce the desired result for (i). The derived equation from the n^{th} impact can be used to express $\mathrm{u_1}$ in terms of $\mathrm{u_0}$ and $\mathrm{v_0}$, after which, using the given solution in the cases $\mathrm{n = 0}$ and $\mathrm{n = 1}$ yields

$\mathrm{A = -17u_0 + 3v_0}$ and $\mathrm{B = 18u_0 - 3v_0}$. Expressing $\mathrm{u_n}$ as

$\left(\frac{5}{7}\right)^{\mathrm{n}} \left[\mathrm{(-17u_0 + 3v_0)}\left(\frac{49}{50}\right)^{\mathrm{n}} + \mathrm{(18u_0 - 3v_0)}\right]$, the final result of the question follows.

Question 10

That $\sin\beta = \frac{m}{M+m}$ can be found by taking moments about O, A, or G (say, the centre of mass of the combined disc and particle). Applying the cosine rule to triangle OAP and using the first result obtained leads directly to the first displayed result. The constant expression is obtained by conserving energy, the first term being the kinetic energy of the disc, the second term being the kinetic energy of the particle and the last term being the potential energy of the combined centre of mass relative to a zero energy level defined by the equilibrium position of G. Differentiating the energy expression with respect to time, substituting for m, I, $\sin\beta$, and $\cos\beta$, the derived equation simplifies to that of an approximate simple harmonic motion, which with the small angle approximation leads to SHM with the required period.

Question 11

Resolving radially for the particle and then setting the tension to zero (as the string becomes slack) yields $\mathrm{V}^2 = \mathrm{bg}\cos\alpha$. Expressing the horizontal and vertical displacements of the particle from O at a time T from the instant the projectile flight commences, and then expressing the sum of the squares of these displacements as the length of the string squared gives an equation which when simplified and the first result is used to eliminate $\mathrm{bg}\cos\alpha$ does give the required equation. $\tan\beta$ can be found by dividing the components of the projectile velocity, employing the result just found to simplify to obtain the desired result. The condition that the particle reaches instantaneous rest is that it is moving radially at the point the string becomes taut, and so $\tan\beta$ is also equal to the division of the displacements found when the time is T. Substituting for V^2 and T gives an equation solely in terms of $\sin\alpha$ and $\cos\alpha$ which can be simplified to a biquadratic in $\sin\alpha$, which in turn can be solved but only the required result is positive.

Question 12

Part (i) relies on appreciating that the desired probability is that at least k numbers are less than or equal to y, and so is the sum of the probabilities that a certain number of numbers are less than or equal to y, each of which is a binomial probability. The first result of (ii) relies on the definition of the binomial coefficient and manipulation of the factorials. The similar expression for $(\mathrm{n}-\mathrm{m})\binom{\mathrm{n}}{\mathrm{m}}$

is $\mathrm{n}\binom{\mathrm{n}-1}{\mathrm{m}}$ which can be arrived at in the same way; an alternative nice method relies on the symmetry of binomial coefficients, namely

$(\mathrm{n}-\mathrm{m})\binom{\mathrm{n}}{\mathrm{m}} = (\mathrm{n}-\mathrm{m})\binom{\mathrm{n}}{\mathrm{n}-\mathrm{m}} = \mathrm{n}\binom{\mathrm{n}-1}{\mathrm{n}-\mathrm{m}-1} = \mathrm{n}\binom{\mathrm{n}-1}{\mathrm{m}}$. The expression (*) is a cdf so differentiating it (by the product rule), appreciating that one term in one of the sums is zero and using the first two results of the part (one in each sum), the two sums are then the same, when one sum is re-indexed, bar one term which is the required answer. As the integral of a pdf is 1, the deduced value of the pdf without the constant is $\frac{1}{\mathrm{n}\binom{\mathrm{n}-1}{\mathrm{k}-1}}$.

Integrating in the usual way to find the expectation in part (iii) gives an expression which is a multiple of the pdf for the case of $\mathrm{n}+1$ numbers and the variable defined as the $\mathrm{k}+1$ th smallest, and the two constant expressions cancel to give $\frac{\mathrm{k}}{\mathrm{n}+1}$.

Question 13

The first result of the question is most easily shown by explicitly expressing $\mathrm{G}(1)$ and $\mathrm{G}(-1)$ in terms of probabilities and then evaluating the RHS of the equation. The pgf of the Poisson distribution is standard bookwork, expressing it as an infinite sum using the Poisson probabilities, and recognising that it is an exponential function. The pgf of Y may be found either by finding k first using the results of the stem and then recognising the power series working from the pgf definition or alternatively $\frac{1}{2}\big(\mathrm{G_X(t)} + \mathrm{G_X(-t)}\big)$. The expectation of Y can be found using the pgf as $\mathrm{G'_Y}(1)$, at which point it only remains to demonstrate that $\tanh\lambda\ < 1$. In part (ii), it is sensible to obtain the pgf of Z first, which can be done directly and recognising the sum of a cos and cosh power series or via $\mathrm{G_Z(t)} = \frac{1}{4}\big(\mathrm{G_X(t)} + \mathrm{G_X(-t)} + \mathrm{G_X(it)} + \mathrm{G_X(-it)}\big)$, or even more neatly

$\mathrm{G_Z(t)} = \frac{1}{2}\big(\mathrm{G_Y(t)} + \mathrm{G_Y(it)}\big)$. $\mathrm{E(Z)} = \mathrm{G'_Z}(1) = \frac{\lambda(\sinh\lambda - \sin\lambda)}{(\cosh\lambda + \cos\lambda)}$ and so, for example, choosing $\lambda = \frac{3\pi}{2}$ it can be shown that this is greater than λ by expressing the hyperbolic functions in terms of exponential functions.