STEP MATHEMATICS 2

2019

Hints and Solutions

Question 1

For the introductory part, first find an equation of the tangent to the curve at the point with $x = a$. An expression can then be found for the y-coordinate of the point on the tangent where $x = p$ and this can easily be shown to be equal to 0 if and only if $g'(a) = 0$.

In part (i), the first result follows by identifying that $g(x) = A(x - q)(x - r)$ allows the first result to be applied. The gradient of the tangent can be found by differentiating $f(x)$ and then the fact that $2a = q + r$ can be used to eliminate a from this expression.

In part (ii) the tangent at the point where $x = c$ is essentially another case of the tangent considered in part (i), so the gradient of this tangent can be deduced easily. By equating the gradients of the two tangents it can be deduced that $q - p = r - q$ (although care needs to be taken to justify the choice of square roots). The equation of the tangent at $x = q$ can also be found and so any other points of intersection between this tangent and the curve can be found. The result then follows easily.

Question 2

From a sketch of the function, it can be seen that the first integral corresponds to an area below the curve and the second integral corresponds to an area to the left of the curve. These two areas make a rectangle, whose area is clearly expressed by the expression on the right of the equation.

In the first part it is clear that the value of $g(0)$ satisfies $\left(g(0)\right)^3 + g(0) = 0$. Clearly $g(0) = 0$ satisfies this, but it is necessary to factorise and then show that the quadratic factor has no other real solutions. The second result can be seen by differentiating and observing that $(3g(t)^2 + 1)$ must be greater than 0.

To evaluate the integral, observe that $g^{-1}(s) = s^3 + s$ and apply the result shown at the start of the question.

For the second part it must be noted that this function does not satisfy the conditions for the initial result to be applied. However, it can be seen that $h(t) = g(t + 2)$.

It therefore follows that $h'(t) > 0$ and the values of $h(0)$ and $h(8)$ can be deduced. By considering a sketch of this function it can be seen how to modify the initial result to apply in this case.

Question 3

The initial result can be shown by considering the four possible combinations of signs for x_1 and x_2. Induction can then be used to prove the more general result.

In part (i)(a) the initial result can then be applied to show that the value of $f(x) - 1$ must be less than or equal to a polynomial in $|x|$. Furthermore, the coefficients must also be less than or equal to A and so the value must be less than or equal to a sum that can be seen to be a geometric sequence.

In part (i)(b) the previous result can be applied in the case $x = \omega$ and, since $|\omega| < 1$ it must be the case that $1 - |\omega| > 0$. Therefore, the inequality can be multiplied by $(1 - |\omega|)$ without changing the direction of the inequality. The required inequality then follows easily.

To show that the inequalities continue to hold if $|\omega| > 1$, observe that $\frac{1}{\omega}$ is a root of the polynomial $g(x) = 1 + a_{n-1}x + \cdots + a_1 x^{n-1} + x^n$, as $g\left(\frac{1}{\omega}\right) = \frac{1}{\omega^n} f(\omega) = 0$. Since $g(x)$ has the same properties as $f(x)$, $\frac{1}{|\omega|}$ must also satisfy (*). It then only remains to consider the case $|\omega| = 1$.

For the final part, observe that division by 135 produces a polynomial that satisfies the conditions specified and so the bounds on the value of ω reduces the cases to be considered to $\omega = \pm 1$ and $\omega = \pm 2$.

Question 4

In the first part, if the expression to be evaluated is multiplied by $\sin\left(\frac{\pi}{9}\right)$, then three applications of the given identity can be used. A similar process can then be used to simplify the first expression in part (ii). For the sum, note that $\tan x$ is the derivative of $-\ln(\cos x)$ and so, the result can be obtained by taking logs of the first result and then differentiating term by term.

For the final part, first change to a finite product (from $k = 1$ to $k = n$) and then take the limit as $n \to \infty$. Note however, that the product in part (ii) started at $k = 0$, so the result in part (ii) needs to be modified before it can be applied.

In the same way, the sum can be modified to start at $k = 0$ (where $k = j - 2$) and then the result of part (ii) can be applied with $x = \frac{\pi}{4}$.

Question 5

In part (i) the values of a for which the sequence is constant can be found by solving the equation $a = f(a)$. The sequence will have period 2 if the equation $a = f(f(a))$ has a solution that is different from those for which the sequence is constant. Although the equation $a = f(f(a))$ is a quartic, it is clear that the values of a for which the sequence is constant will be solutions of this equation as well. This means that two factors of the quartic are known and so the remaining factor will be a quadratic. When considering the roots of this quadratic it must also be checked to confirm that the roots are not repeats of the values that give a constant sequence.

In part (ii), note that there cannot be a solution to the equation $f(a) = a$ and so it must be the case that either $f(x) > x$ for all x or $f(x) < x$ for all x (since f is a continuous function). It is clear that $f(x) > x$ for large values of x.

Since it must be that case that $f(x) > x$ for all x if the sequence is not constant, it must also be the case that $f\big(f(x)\big) > x$ for all x.

Finally, it can be seen that, in the case where $q = p$ the sequence is of the form in part (i) and so it should be possible to deduce a case in which there is no value of a for which the sequence has period 2, but there is a value of a for which the sequence is constant.

Question 6

In the first part, substituting $y = mx + c$ into the differential equation will allow the values of m and c to be deduced. Since stationary points must satisfy $\frac{dy}{dx} = 0$, substituting this into the differential equation shows that stationary points must lie on the given line. It then follows that solution curves with $k < 2$ cannot have stationary points as they would have to cross the straight-line solution that has already been found.

Given that the relationship between x and y for any stationary point is known, it is possible to differentiate the differential equation and evaluate $\frac{d^2y}{dx^2}$ for any stationary point.

Once the substitution provided has been applied, the new differential equation can be solved by separating the variables and have equations that can be sketched easily.

In the second part, the same approach as part (i) can be used to find the possible sets of values for m and c. The RHS of the differential equation can be considered a function of $y - x$ and this allows it to be factorised. Solving $\frac{dy}{dx} = 0$ then shows that x and y must satisfy one of two linear equations and the sign of $\frac{dy}{dx}$ can be deduced for points between these two lines.

The graph can then be sketched, remembering that the curve cannot cross the two straight-line solutions.

Question 7

In part (i), taking the scalar product of $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ with each of the vectors in turn produces a set of three equations from which it can be deduced that $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{a}$ and that any pair of them add up to -1. Alternatively, it can be observed that $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = (-\boldsymbol{c}) \cdot (-\boldsymbol{c})$.

It can then be shown that the angle between any pair of these vectors is $120°$ and so a sketch shows that the triangle must be equilateral.

In part (ii), a similar approach will lead to the given result. Alternatively, the result can be obtained by observing that $(\boldsymbol{a_1} + \boldsymbol{a_2}) \cdot (\boldsymbol{a_1} + \boldsymbol{a_2}) = (-\boldsymbol{a_3} - \boldsymbol{a_4}) \cdot (-\boldsymbol{a_3} - \boldsymbol{a_4})$. For part (a), note that it must be the case that the angle between any pair of vectors is equal to the angle between the other two vectors.

For part (b) use the vector $(\boldsymbol{a_1} - \boldsymbol{a_2})$ to find the length of one side of the tetrahedron. From the fact that the tetrahedron is regular it can be deduced that $\boldsymbol{a_1} \cdot \boldsymbol{a_2} = \boldsymbol{a_1} \cdot \boldsymbol{a_3} = \boldsymbol{a_1} \cdot \boldsymbol{a_4}$. The side length can then be calculated.

Question 8

In part (i), the property of f means that $f(\boldsymbol{M}) = f(\boldsymbol{MI}) = f(\boldsymbol{M})f(\boldsymbol{I})$. Note that justification of $f(\boldsymbol{I}) = 1$ requires that $f(\boldsymbol{M}) \neq 0$.

In part (ii), note that $\boldsymbol{J}^2 = \boldsymbol{I}$ and so the value of $f(\boldsymbol{J})$ must be either 1 or -1. The second result of this part follows from application of the property of function f.

For part (iii), first show that $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0$ by applying the result of part (ii) and then pre-multiply this matrix by $\boldsymbol{K}$ to obtain one in which the second row is a multiple of the first.

For part (iv), note that $\boldsymbol{P}^2 = \boldsymbol{K}^{-1}\boldsymbol{PK}$ in the case where $k = 2$. This leads to the fact that $f(\boldsymbol{P})$ must be either 0 or 1. The fact that $\boldsymbol{P}^{-1}$ exists can then be used to show that $f(\boldsymbol{P})$ cannot be 0.

Question 9

In part (i), the position vector of the particle at time t can be calculated. The distance OP is then the modulus of this vector. It is easier to differentiate the square of the distance with respect to time (which is sufficient as this will be increasing if and only if the distance is increasing). The resulting expression can be shown to be positive if $\sin\alpha < \frac{2\sqrt{2}}{3}$. Similarly, in the case where $\sin\alpha > \frac{2\sqrt{2}}{3}$ it is possible to identify a value of t for which the distance is certainly decreasing and show that this is before the moment at which the particle lands.

In part (ii), the vector QP can again be calculated and then the distance PQ found by taking the modulus. As in part (i) it is simpler to deal with PQ^2 rather than PQ. In this case, care must be taken with the inequality to check that both sides are positive before they are squared and used to justify that the distance is increasing throughout the flight of P.

Question 10

A diagram is very useful in this question. First, note that the triangle ABC must be isosceles and then take moments about A. In the case given in part (i) this then shows that $T > W$ and so the string will break.

In part (ii), resolve the forces vertically to find an expression for the reaction force and then this can be used to find an expression for the maximum possible value for the frictional force. W can then be eliminated using the equation in part (i) found by taking moments about A. Rearranging then leads to an expression that can be used to explain the required result.

For the third part, the values of k for which breaking and slipping occur can be found from the answers to part (i). These two values can be used to set up an inequality that must be satisfied in order for slipping to occur before the string breaks.

Question 11

In part (i), the numbers of ways of choosing the pairs can be found by checking the numbers of possible values for n_2 for each choice of n_1. A clear list of the possibilities for each case should then make generalised formulae for the cases $n_3 = 2n + 1$ and $n_3 = 2n$.

In part (ii), the possible combinations which lead to a triangle match those found in the first part of the question. There are $\binom{N-1}{2}$ possibilities for the shorter two rods if the length of the longest rod is known, so combining this with the answers to part (i) the probability can be calculated for each of the two cases to be considered.

In part (iii), the probability can be calculated by multiplying the probability in part (ii) for each possible length of the longest rod by the probability that that length is the longest of the three rods. Adding all of these together will result in the overall probability that the rods can form a triangle.

Question 12

Part (i) requires a simple integration to calculate the values of $E(X)$ and $E(X^2)$. The required result then follows algebraically.

In part (ii), use integration to find the values of the quartiles and hence the interquartile range. Square the two values to allow them to be compared with each other.

In part (iii), the binomial expansion should be easy to write down, but note that the $(k+1)^{th}$ term is the term in x^k, not x^{k+1}. The lower quartile and median can be evaluated by integration of $f(x)$. To show the inequalities, note that $\left(\frac{1}{\mu}\right)^n = \left(1+\frac{1}{n}\right)^n$ and that each term in the expansion is positive, so the value must be greater than the sum of the first two terms. Similarly, the $(k+1)^{th}$ term of the expansion can be shown to be greater than $\frac{1}{k!}$, so the result that may be assumed will lead to the other inequality.