STEP MATHEMATICS 3

2021

Mark Scheme

1. (i) $x = -4\cos^3 t$ so $\frac{dx}{dt} = 12\cos^2 t\ \sin t$ **M1**

$y = 12\sin t - 4\ \sin^3 t$ so $\frac{dy}{dt} = 12\cos t - 12\sin^2 t\ \cos t = 12\cos t\ (1 - \sin^2 t) = 12\cos^3 t$

M1

So $\frac{dy}{dx} = \frac{12\cos^3 t}{12\cos^2 t\ \sin t} = \cot t$ **A1**

Thus the equation of the normal at $(-4\cos^3\varphi\ , 12\sin\varphi - 4\ \sin^3\varphi\ )$ is

$$y - (12\sin\varphi - 4\ \sin^3\varphi) = -\frac{1}{\cot\varphi}(x - -4\cos^3\varphi)$$

M1 A1ft

This simplifies to $x\sin\varphi + y\cos\varphi = 12\sin\varphi\cos\varphi - 4\sin^3\varphi\cos\varphi - 4\sin\varphi\cos^3\varphi$

That is $x\sin\varphi + y\cos\varphi = 8\sin\varphi\cos\varphi$ **A1 (6)**

Alternative simplification $x\tan\varphi + y = 8\sin\varphi$

For $x = 8\cos^3 t$, $\frac{dx}{dt} = -24\cos^2 t\ \sin t$ and for $y = 8\ \sin^3 t$, $\frac{dy}{dt} = 24\sin^2 t\ \cos t$

So $\frac{dy}{dx} = \frac{24\sin^2 t\ \cos t}{-24\cos^2 t\ \sin t} = -\tan t$ **M1 A1ft**

Thus the equation of the tangent to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at $(8\cos^3\varphi, 8\ \sin^3\varphi\ )$ is

$$y - 8\ \sin^3\varphi = -\tan\varphi\ (x - 8\cos^3\varphi)$$

M1

This simplifies to

$x\sin\varphi + y\cos\varphi = 8\ \sin^3\varphi\cos\varphi +\ 8\sin\varphi\cos^3\varphi = 8\sin\varphi\cos\varphi\ (\sin^2\varphi + \cos^2\varphi)$

That is $x\sin\varphi + y\cos\varphi = 8\sin\varphi\cos\varphi$ as required. **A1 (4)**

Alternative 1

the normal is a tangent to the second curve if it has the same gradient and the point $(8\cos^3\varphi, 8\ \sin^3\varphi\ )$ lies on the normal. **M1**

Gradient working as before **M1A1ft**

Substitution $x\sin\varphi + y\cos\varphi = 8\sin\varphi\ \cos^3\varphi + 8\ \sin^3\varphi\ \cos\varphi = 8\sin\varphi\cos\varphi\,(\sin^2\varphi + \cos^2\varphi) = 8\sin\varphi\cos\varphi$ as required or $x\tan\varphi + y = 8\sin\varphi\cos\varphi\,(\sin^2\varphi + \cos^2\varphi)$ **A1**

Alternative 2

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

M1

(ii) $x = \cos t + t\sin t$ so $\frac{dx}{dt} = -\sin t + t\cos t + \sin t = t\cos t$

$y = \sin t - t\ \cos t$ so $\frac{dy}{dt} = \ \cos t - \cos t + t\sin t = t\sin t$ **M1**

So $\frac{dy}{dx} = \tan t$ **A1**

Thus the equation of the normal at $(\cos\varphi + \ \varphi\sin\ \varphi\ ,\ \sin\ \varphi - \ \varphi\ \cos\ \varphi\,)$ is

$$y - (\sin\ \varphi - \ \varphi\ \cos\ \varphi) = -\cot\varphi\big(x - (\cos\varphi + \ \varphi\sin\ \varphi)\big)$$

M1 A1ft

This simplifies to $x\cos\varphi + y\ \sin\ \varphi = 1$ **A1 (5)**

Alternatives which can be followed through to perpendicular distance step, or alternative method # are

$x + y\ \tan\varphi = \sec\varphi$ and $x\cot\varphi + y\ = \csc\varphi$

The distance of $(0{,}0)$ from $x\cos\varphi + y\ \sin\ \varphi = 1$ is $\left|\frac{-1}{\sqrt{\cos^2\varphi + \sin^2\varphi}}\right| = 1$

M1 A1ft **A1**

Alternatively, the perpendicular to $x\cos\varphi + y\ \sin\ \varphi = 1$ through $(0{,}0)$ is

$y\cos\varphi - x\ \sin\ \varphi = 0$, and these two lines meet at $(\cos\varphi\ ,\ \sin\ \varphi\,)$

M1 A1ft

which is a distance $\sqrt{\cos^2\varphi + \sin^2\varphi} = 1$ from $(0{,}0)$. **A1**

So the curve to which this normal is a tangent is a circle centre $(0{,}0)$, radius 1 which is thus $x^2 + y^2 = 1$ **M1 A1 (5)**

2. (i) $\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-(b-c)x \\ b-(c-a)y \\ c-(a-b)z \end{pmatrix} = \begin{pmatrix} a-a \\ b-b \\ c-c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as required. **M1 A1***

As a, b and c are distinct, they cannot all be zero. If M^{-1} exists $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which is a contradiction.

So, M^{-1} does not exist and thus $det\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} = 0$, **M1**

i.e. $1 - xyz + xyz + yz + zx + xy = 0$, (Sarus)

or $1(1+yz) - -x(y-yz) + x(yz+z) = 0$ (by co-factors) **M1**

which simplifies to

$yz + zx + xy = -1$ **A1 * (5)**

$(x+y+z)^2 \geq 0$

So $x^2 + y^2 + z^2 + 2yz + 2zx + 2xy \geq 0$ **M1**

and so $x^2 + y^2 + z^2 \geq 2$ **A1* (2)**

(ii) $\begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a-(b+c)x \\ 2b-(c+a)y \\ 2c-(a+b)z \end{pmatrix} = \begin{pmatrix} 2a-2a \\ 2b-2b \\ 2c-2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

B1 **M1** **A1**

As a, b and c are positive, they cannot all be zero. Thus as $\begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

as in part (i), $det\begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix} = 0$,

i.e. $8 - xyz - xyz - 2yz - 2zx - 2xy = 0$, that is **M1 A1**

$xyz + yz + zx + xy = 4$ **A1* (6)**

$$(x+1)(y+1)(z+1) = xyz + zx + xy + x + y + z + 1 = 4 + x + y + z + 1 > 5$$

M1 **A1**

because as a, b, and c are all positive, so are x, y and z. **E1**

Thus $\left(\frac{2a}{b+c}+1\right)\left(\frac{2b}{c+a}+1\right)\left(\frac{2c}{a+b}+1\right) > 5$

Multiplying by $(b+c)(c+a)(a+b)$, all three factors of which are positive, gives

$(2a+b+c)(a+2b+c)(a+b+c) > 5(b+c)(c+a)(a+b)$ as required. **A1* (4)**

$x = \frac{2a}{b+c} > \frac{2a}{a+b+c}$ as a, b, and c are positive, and similarly both, $y > \frac{2b}{a+b+c}$ and $z > \frac{2c}{a+b+c}$

M1

Thus $4 + x + y + z + 1 > 4 + \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2c}{a+b+c} + 1 = 4 + \frac{2(a+b+c)}{a+b+c} + 1 = 7$

dM1

and thus following the argument used to obtain the previous result

$(2a+b+c)(a+2b+c)(a+b+c) > 7(b+c)(c+a)(a+b)$ as required.

A1* (3)

3. (i)

$$\frac{1}{2}\left(I_{n+1}+I_{n-1}\right)=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n+1}+(\sec x+\tan x)^{n-1}dx$$

$$=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left((\sec x+\tan x)^2+1\right)dx$$

M1

$$=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left(\sec^2 x+2\sec x\tan x+\tan^2 x+1\right)dx$$

$$=\int_0^\beta (\sec x+\tan x)^{n-1}(\sec^2 x+\sec x\tan x)\,dx$$

M1

$$=\left[\frac{1}{n}(\sec x+\tan x)^n\right]_0^\beta=\frac{1}{n}\left((\sec\beta+\tan\beta)^n-1\right)$$

M1 A1 ***A1 (5)**

as required.

$$\frac{1}{2}\left(I_{n+1}+I_{n-1}\right)-I_n=\frac{1}{2}\left(I_{n+1}-2I_n+I_{n-1}\right)$$

$$=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n+1}-2(\sec x+\tan x)^n+(\sec x+\tan x)^{n-1}dx$$

M1

$$=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left((\sec x+\tan x)-1\right)^2dx$$

M1 A1

$\left((\sec x+\tan x)-1\right)^2>0$ for all x>0

$\sec x\geq 1$ for $0\leq x<\frac{\pi}{2}$ and hence for $0\leq \mathrm{x}<\beta$ and similarly $\tan x\geq 0$, and thus also $(\sec x+\tan x)^{n-1}>0$. **E1**

Therefore, $\frac{1}{2}\left(I_{n+1}+I_{n-1}\right)-I_n>0$, **A1**

and so $I_n<\frac{1}{2}\left(I_{n+1}+I_{n-1}\right)=\frac{1}{n}\left((\sec\beta+\tan\beta)^n-1\right)$ as required. **M1 *A1 (7)**

Alternative 1: it has already been shown that

$$\frac{1}{2}(I_{n+1}+I_{n-1})=\int_0^\beta (\sec x+\tan x)^{n-1}(\sec^2 x+\sec x\tan x)\,dx$$

$$=\int_0^\beta \sec x\,(\sec x+\tan x)^n\,dx$$

which is greater than I_n as the expression being integrated is greater than $(\sec x+\tan x)^n$ because $\sec x>0$ over this domain.

Alternative 2:-

$$I_{n+1}-I_n=\int_0^\beta (\sec x+\tan x)^n(\sec x+\tan x-1)\,dx$$

$$I_n-I_{n-1}=\int_0^\beta (\sec x+\tan x)^{n-1}(\sec x+\tan x-1)\,dx$$

M1 A1 A1

For $0<\mathrm{x}<\beta$, $\sec x>1$, $\tan x>0$ so $\sec x+\tan x>1$ **E1** and thus $I_{n+1}-I_n>I_n-I_{n-1}$ **A1**

and so $I_n\le\frac{1}{2}(I_{n+1}+I_{n-1})=\frac{1}{n}((\sec\beta+\tan\beta)^n-1)$ **M1 *A1 (7)**

(ii) $\frac{1}{2}(J_{n+1}+J_{n-1})=\frac{1}{2}\int_0^\beta(\sec x\cos\beta+\tan x)^{n+1}+(\sec x\cos\beta+\tan x)^{n-1}dx$

$$=\frac{1}{2}\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left((\sec x\cos\beta+\tan x)^2+1\right)dx$$

M1

$$=\frac{1}{2}\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x\cos^2\beta+2\sec x\cos\beta\tan x+\tan^2 x+1\right)dx$$

$$=\frac{1}{2}\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x\,(1-\sin^2\beta)+2\sec x\cos\beta\tan x+\tan^2 x+1\right)dx$$

$$=\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left((\sec^2 x+\sec x\cos\beta\tan x)-\sec^2 x\sin^2\beta\right)dx$$

M1

$$\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x+\sec x\cos\beta\tan x\right)dx=\left[\frac{1}{n}(\sec x\cos\beta+\tan x)^n\right]_0^\beta$$

M1

$$= \frac{1}{n}((1 + \tan\beta)^n - \cos^n\beta)$$

A1

$$\int_0^\beta (\sec x \cos\beta + \tan x)^{n-1}\ \sec^2 x \sin^2\beta\, dx > 0$$

by a similar argument to part (i), namely $\sec^2 x \sin^2\beta > 0$ for any x, and $\sec x \cos\beta + \tan x > 0$ as $\sec x > 0$ and $\tan x \geq 0$ for $0 \leq \mathrm{x} < \beta < \frac{\pi}{2}$ **E1**

Hence $\frac{1}{2}\,(J_{n+1} + J_{n-1}) < \frac{1}{n}((1 + \tan\beta)^n - \cos^n\beta)$ **A1**

But

$$\frac{1}{2}\,(J_{n+1} + J_{n-1}) - J_n == \frac{1}{2}\int_0^\beta (\sec x \cos\beta + \tan x)^{n-1}\left((\sec x \cos\beta + \tan x) - 1\right)^2 dx > 0$$

M1

as before, and thus $J_n < \frac{1}{2}\,(J_{n+1} + J_{n-1}) < \frac{1}{n}((1 + \tan\beta)^n - \cos^n\beta)$ as required. ***A1 (8)**

4. (i)

$\boldsymbol{m}.\boldsymbol{a} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{a} = \frac{1}{2}(1+\boldsymbol{a}.\boldsymbol{b}) = m\cos\alpha$ where α is the non-reflex angle between **a** and **m**

$\boldsymbol{m}.\boldsymbol{b} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{b} = \frac{1}{2}(1+\boldsymbol{a}.\boldsymbol{b}) = m\cos\beta$ where α is the non-reflex angle between $\boldsymbol{b}$ and **m**

M1 A1

Thus $\cos\alpha = \cos\beta$ and so $\alpha = \beta$ as for $0 \le \tau \le \pi$, there is only one value of τ for any given value of $\cos\tau$. **E1 (3)**

(ii) $\boldsymbol{a_1}.\boldsymbol{c} = (\boldsymbol{a} - (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}).\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{c}\,\boldsymbol{c}.\boldsymbol{c} = 0$ as required. ***B1**

$\boldsymbol{a}.\boldsymbol{c} = \cos\alpha$, $\boldsymbol{b}.\boldsymbol{c} = \cos\beta$, $\boldsymbol{a}.\boldsymbol{b} = \cos\theta$

$\boldsymbol{a_1} = \boldsymbol{a} - (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}$ and $\boldsymbol{b_1} = \boldsymbol{b} - (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{c}$

$$|\boldsymbol{a_1}|^2 = \boldsymbol{a_1}.\boldsymbol{a_1} = (\boldsymbol{a} - (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}).(\boldsymbol{a} - (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{c}\,\boldsymbol{a}.\boldsymbol{c} + \boldsymbol{a}.\boldsymbol{c}\,\boldsymbol{a}.\boldsymbol{c}\,\boldsymbol{c}.\boldsymbol{c}$$

$$= 1 - 2\cos^2\alpha + \cos^2\alpha = \sin^2\alpha$$

M1

and so. as α is acute, $|\boldsymbol{a_1}| = \sin\alpha$ as required. ***A1**

$$\boldsymbol{a_1}.\boldsymbol{b_1} = (\boldsymbol{a} - (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}).(\boldsymbol{b} - (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} - 2(\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{b}.\boldsymbol{c}) + (\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{b}.\boldsymbol{c})(\boldsymbol{c}.\boldsymbol{c})$$

$$= \cos\theta - \cos\alpha\cos\beta$$

M1 A1

but also, $\boldsymbol{a_1}.\boldsymbol{b_1} = \sin\alpha\sin\beta\cos\varphi$ **B1 M1**

and hence,

$$\cos\varphi = \frac{\cos\theta - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

as required. ***A1 (8)**

(iii) $\boldsymbol{m_1} = \boldsymbol{m} - (\boldsymbol{m}.\boldsymbol{c})\boldsymbol{c} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}) - \left(\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{c}\right)\boldsymbol{c} = \frac{1}{2}(\boldsymbol{a_1}+\boldsymbol{b_1})$ **B1**

$\boldsymbol{m_1}$ bisects the angle between $\boldsymbol{a_1}$ and $\boldsymbol{b_1}$ if and only if

$$\frac{\boldsymbol{m_1}.\boldsymbol{a_1}}{\sin\alpha} = \frac{\boldsymbol{m_1}.\boldsymbol{b_1}}{\sin\beta}$$

M1

Thus, multiplying through by $2\sin\alpha\sin\beta$,

$$(\boldsymbol{a_1}+\boldsymbol{b_1}).\boldsymbol{a_1}\sin\beta = (\boldsymbol{a_1}+\boldsymbol{b_1}).\boldsymbol{b_1}\sin\alpha$$

A1

$$(\sin^2\alpha + \boldsymbol{a_1}.\boldsymbol{b_1})\sin\beta = (\sin^2\beta + \boldsymbol{a_1}.\boldsymbol{b_1})\sin\alpha$$

M1 A1

So

$$(\boldsymbol{a_1}.\boldsymbol{b_1} - \sin\alpha\sin\beta)(\sin\alpha - \sin\beta) = 0$$

A1

and thus, $\sin\alpha = \sin\beta$ in which case $\alpha = \beta$ as both angles are acute, ***A1**

or $\cos\theta - \cos\alpha\cos\beta = \sin\alpha\sin\beta$, meaning that $\cos\theta = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

M1 *A1 (9)

5. (i) The curves meet when $a + 2\cos\theta = 2 + \cos 2\theta$

That is, $a + 2\cos\theta = 2 + 2\cos^2\theta - 1$ or as required, **B1** $2\cos^2\theta - 2\cos\theta + 1 - a = 0$

The curves touch if this quadratic has coincident roots, **M1** i.e. if $4 - 8(1-a) = 0 \Rightarrow a = \frac{1}{2}$, ***A1** or if $\cos\theta = \pm 1$, **M1** in which cases $a = 1$ **A1** or $a = 5$. **A1 (6)**

Alternatively, for the curves to touch, they must have the same gradient, so differentiating,

$$-2\sin\theta = -2\sin 2\theta = -4\sin\theta\,\cos\theta$$

M1

in which case, either $\sin\theta = 0$ giving $\cos\theta = \pm 1$, **M1** in which cases $a = 1$ **A1** or $a = 5$, **A1** or $\cos\theta = \frac{1}{2}$ in which case $a = \frac{1}{2}$. ***A1 (6)**

(ii) If $a = \frac{1}{2}$ then at points where they touch, $\cos\theta = \frac{1}{2}$ so $\theta = \pm\frac{\pi}{3}$ and thus $\left(\frac{3}{2}, \pm\frac{\pi}{3}\right)$. **M1A1**

$r = a + 2\cos\theta$ is symmetrical about the initial line which it intercepts at $\left(\frac{5}{2}, 0\right)$ and has a cusp at $\left(0, \pm\cos^{-1}\left(-\frac{1}{4}\right)\right)$. It passes through $\left(\frac{1}{2}, \pm\frac{\pi}{2}\right)$ and only exists for

$-\cos^{-1}\left(-\frac{1}{4}\right) < \theta < \cos^{-1}\left(-\frac{1}{4}\right)$.

$r = 2 + \cos 2\theta$ is symmetrical about both the initial line, and its perpendicular. It passes through

$(3,0)$, $(3, \pi)$, and $\left(1, \pm\frac{\pi}{2}\right)$

Sketch **G6 (8)**

(iii) If $a = 1$, then the curves meet where $2\cos^2\theta - 2\cos\theta = 0$, i.e. $\cos\theta = 1$ at $(3,0)$ where they touch, and $\cos\theta = 0$ at $\left(1, \pm\frac{\pi}{2}\right)$

$r = a + 2\cos\theta$ is symmetrical about the initial line which it intercepts at $(3, 0)$ and has a cusp at $\left(0, \pm\cos^{-1}\left(-\frac{1}{2}\right)\right) = \left(0, \pm\frac{2\pi}{3}\right)$. It passes through $\left(1, \pm\frac{\pi}{2}\right)$ and only exists for

$-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$.

Sketch **G3**

If $a = 5$, then the curves meet where $2\cos^2\theta - 2\cos\theta - 4 = 0$, i.e. only $\cos\theta = -1$ at $(3, \pi)$ where they touch, as $\cos\theta \neq 2$.

$r = a + 2\cos\theta$ is symmetrical about the initial line which it intercepts at $(7, 0)$ and $(3, \pi)$. It also passes through $\left(5, \pm\frac{\pi}{2}\right)$.

Sketch **G3 (6)**

6. (i)

$$f_\alpha(x) = \tan^{-1}\left(\frac{x\tan\alpha + 1}{\tan\alpha - x}\right)$$

$$f'_\alpha(x) = \frac{1}{1+\left(\frac{x\tan\alpha+1}{\tan\alpha - x}\right)^2}\frac{(\tan\alpha - x)\tan\alpha + (x\tan\alpha + 1)}{(\tan\alpha - x)^2}$$

M1 A1

$$= \frac{\tan^2\alpha + 1}{(\tan\alpha - x)^2 + (x\tan\alpha + 1)^2}$$

$$= \frac{\sec^2\alpha}{\tan^2\alpha + x^2 + x^2\tan^2\alpha + 1} = \frac{\sec^2\alpha}{\sec^2\alpha\,(1+x^2)} = \frac{1}{1+x^2}$$

M1 **M1** ***A1 (5)**

as required.

Alternative

$$f_\alpha(x) = \tan^{-1}\left(\frac{x\tan\alpha + 1}{\tan\alpha - x}\right)$$

$$= \tan^{-1}\left(\frac{x + \cot\alpha}{1 - x\cot\alpha}\right)$$

$$= \tan^{-1}\left(\frac{\tan(\tan^{-1}x) + \tan\left(\frac{\pi}{2} - \alpha\right)}{1 - \tan(\tan^{-1}x)\tan\left(\frac{\pi}{2}-\alpha\right)}\right)$$

M1 A1

$$= \tan^{-1}\left(\tan\left(\tan^{-1}x + \frac{\pi}{2} - \alpha\right)\right)$$

M1

$= \tan^{-1}x + \frac{\pi}{2} - \alpha$ if this is less than $\frac{\pi}{2}$, i.e. if $x < \tan\alpha$

or $= \tan^{-1}x - \frac{\pi}{2} - \alpha$ if $x > \tan\alpha$ **M1**

So $f'_\alpha(x) = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ ***A1 (5)**

Thus $f_\alpha(x) = \tan^{-1}x + c$

$$f_\alpha(0) = \tan^{-1}\left(\frac{1}{\tan\alpha}\right) = \tan^{-1}(\cot\alpha) = \frac{\pi}{2} - \alpha$$

$f_\alpha(x) = 0$ when $x = -\cot\alpha$

There is a discontinuity at $x = \tan\alpha$, with $f_\alpha(x)$ approaching $\frac{\pi}{2}$ from below and $-\frac{\pi}{2}$ from above.

As $x \to \pm\infty$, $f_\alpha(x) \to \tan^{-1}(-\tan\alpha) = -\alpha$

So $f_\alpha(x) = \tan^{-1} x + \frac{\pi}{2} - \alpha$ for $x < \tan\alpha$ and $f_\alpha(x) = \tan^{-1} x - \frac{\pi}{2} - \alpha$ for $x > \tan\alpha$

Sketch **G1 G1 G1 (3)**

$$y = f_\alpha(x) - f_\beta(x) =$$

$\left(\frac{\pi}{2} - \alpha\right) - \left(\frac{\pi}{2} - \beta\right) = \beta - \alpha$ for $x < \tan\alpha$

$\left(-\frac{\pi}{2} - \alpha\right) - \left(\frac{\pi}{2} - \beta\right) = \beta - \alpha - \pi$ for $\tan\alpha < x < \tan\beta$

and $\left(-\frac{\pi}{2} - \alpha\right) - \left(-\frac{\pi}{2} - \beta\right) = \beta - \alpha$ for $x > \tan\beta$

Sketch **G1 G1 G1 (3)**

(ii) $g(x) = \tanh^{-1}(\sin x) - \sinh^{-1}(\tan x)$

$$g'(x) = \frac{1}{1 - \sin^2 x}\cos x - \frac{1}{\sqrt{1 + \tan^2 x}}\sec^2 x$$

M1 A1 **A1**

$$= \frac{\cos x}{\cos^2 x} - \frac{\sec^2 x}{|\sec x|} = \sec x - \frac{\sec^2 x}{-\sec x} = 2\sec x$$

M1 ***A1 (5)**

as required, for $\sec x < 0$, i.e. for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

(For $\sec x > 0$, $g'(x) = 0$)

Sketch **G1 G1 G1 G1 (4)**

7.

$$z = \frac{e^{i\theta} + e^{i\varphi}}{e^{i\theta} - e^{i\varphi}}$$

$$= \frac{\cos\theta + i\sin\theta + \cos\varphi + i\sin\varphi}{\cos\theta + i\sin\theta - \cos\varphi - i\sin\varphi}$$

M1

$$= \frac{2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} + 2i\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}}{-2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2} + 2i\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}}$$

M1 A1 A1

$$= \frac{2\cos\frac{\theta-\varphi}{2}\left(\cos\frac{\theta+\varphi}{2} + i\sin\frac{\theta+\varphi}{2}\right)}{2\sin\frac{\theta-\varphi}{2}\left(i\cos\frac{\theta+\varphi}{2} - \sin\frac{\theta+\varphi}{2}\right)}$$

$$= -i\cot\frac{\theta-\varphi}{2}$$

$$= i\cot\frac{\varphi-\theta}{2}$$

***A1 (5)**

as required.

Alternatively,

$$z = \frac{e^{i\theta} + e^{i\varphi}}{e^{i\theta} - e^{i\varphi}} = \frac{e^{i\left(\frac{\theta-\varphi}{2}\right)} + e^{-i\left(\frac{\theta-\varphi}{2}\right)}}{e^{i\left(\frac{\theta-\varphi}{2}\right)} - e^{-i\left(\frac{\theta-\varphi}{2}\right)}} = \frac{2\cos\frac{\theta-\varphi}{2}}{2i\sin\frac{\theta-\varphi}{2}} = -i\cot\frac{\theta-\varphi}{2} = i\cot\frac{\varphi-\theta}{2}$$

M1 **M1 A1 A1** ***A1 (5)**

$$|z| = \left|\cot\frac{\theta-\varphi}{2}\right|$$

M1 A1

$$|\arg z| = \frac{\pi}{2}$$

[or $\arg z = \frac{\pi}{2}$ or $\frac{3\pi}{2}$]

M1 A1 (4)

(ii) Let $a = e^{i\alpha}$ and $b = e^{i\beta}$ **M1** then $x = a + b = e^{i\alpha} + e^{i\beta}$ and $AB = b - a = e^{i\beta} - e^{i\alpha}$

$$\arg x - \arg AB = \arg\frac{x}{AB} = \arg\frac{e^{i\alpha} + e^{i\beta}}{e^{i\beta} - e^{i\alpha}}$$

so using (i), $|\arg x - \arg AB| = \frac{\pi}{2}$ **A1** and thus OX and AB are perpendicular, since $x = a + b \neq 0$ and $a \neq b$ as A and B are distinct. **E1 (3)**

Alternative:- $0, a, a+b, b$ define a rhombus OAXB as $|a| = |b| = 1$. Diagonals of a rhombus are perpendicular (and bisect one another).

(iii) $h = a + b + c$ so $AH = a + b + c - a = b + c$ and $BC = c - b$ and thus

$$\frac{AH}{BC} = \frac{b+c}{c-b}$$

B1

as $c - b \neq 0$

From (ii),

$$\left|\arg\frac{AH}{BC}\right| = \frac{\pi}{2}$$

so BC is perpendicular to AH **E1**

unless $b + c = 0$ **E1** in which case $h = a$ **E1 (4)**

(iv) $p = a + b + c \quad q = b + c + d \quad r = c + d + a \quad s = d + a + b$

The midpoint of AQ is $\frac{a+q}{2} = \frac{a+b+c+d}{2}$ and so by its symmetry it is also the midpoint of BR, CS, and DP, **B1 E1**

and thus ABCD is transformed to PQRS by a rotation of π radians about midpoint of AQ. **E1 B1 (4)**

Alternatively, ABCD is transformed to PQRS by an enlargement scale factor -1 , centre of enlargement midpoint of AQ.

8. (i) Suppose $x_k \geq 2 + 4^{k-1}(a-2)$ for some particular integer k (and this is positive as $a > 2$)

E1

Then $x_{k+1} = {x_k}^2 - 2 \geq [2 + 4^{k-1}(a-2)]^2 - 2 = 4 + 4^k(a-2) + 4^{2k-2}(a-2)^2 - 2$

$$= 2 + 4^k(a-2) + 4^{2k-2}(a-2)^2$$

$$> 2 + 4^k(a-2)$$

M1 A1

which is the required result for $k+1$.

For $n = 1$, $2 + 4^{n-1}(a-2) = 2 + a - 2 = a$ so in this case, $x_n = 2 + 4^{n-1}(a-2)$ **B1** and thus by induction $x_n \geq 2 + 4^{n-1}(a-2)$ for positive integer n. **E1 (5)**

(ii) If $|x_k| \leq 2$, then $0 \leq |x_k|^2 \leq 4$, so $-2 \leq |x_k|^2 - 2 \leq 2$, that is $-2 \leq x_{k+1} \leq 2$. **M1A1**

If $|a| \leq 2$, $|x_1| \leq 2$ and thus by induction $-2 \leq x_n \leq 2$, that is $x_n \nrightarrow \infty$ **E1**

Whether $a = \pm\alpha$, x_2 would equal the same value, namely $\alpha^2 - 2$. **E1**

So to consider $|a| \geq 2$, we only need consider $a > 2$ to discuss the behaviour of all terms after the first. Therefore, from part (i), we know $x_n \geq 2 + 4^{n-1}(|a|-2)$ for $n \geq 2$, and thus $x_n \to \infty$ as $n \to \infty$; **B1** hence we have shown $x_n \to \infty$ as $n \to \infty$ if and only if $|a| \geq 2$. **(5)**

(iii)

$$y_k = \frac{Ax_1x_2\cdots x_k}{x_{k+1}}$$

$$y_{k+1} = \frac{Ax_1x_2\cdots x_{k+1}}{x_{k+2}} = \frac{{x_{k+1}}^2}{x_{k+2}}y_k$$

M1

Suppose that

$$y_k = \frac{\sqrt{{x_{k+1}}^2 - 4}}{x_{k+1}}$$

for some positive integer k, **E1** then

$$y_{k+1} = \frac{{x_{k+1}}^2}{x_{k+2}}\frac{\sqrt{{x_{k+1}}^2 - 4}}{x_{k+1}} = \frac{x_{k+1}\sqrt{{x_{k+1}}^2 - 4}}{x_{k+2}}$$

As $x_{k+2} = {x_{k+1}}^2 - 2$, $x_{k+1} = \sqrt{x_{k+2} + 2}$, and $\sqrt{{x_{k+1}}^2 - 4} = \sqrt{x_{k+2} - 2}$,

and thus,

$$y_{k+1} = \frac{\sqrt{x_{k+2} + 2}\sqrt{x_{k+2} - 2}}{x_{k+2}} = \frac{\sqrt{{x_{k+2}}^2 - 4}}{x_{k+2}}$$

M1 A1

which is the required result for $k+1$.

$$y_1 = \frac{Ax_1}{x_2}$$

and also we wish to have

$$y_1 = \frac{\sqrt{{x_2}^2 - 4}}{x_2}$$

M1

then $Ax_1 = \sqrt{{x_2}^2 - 4}$, that is $A^2{x_1}^2 = {x_2}^2 - 4$, and as $x_1 = a$, $x_2 = {x_1}^2 - 2 = a^2 - 2$

so

$A^2a^2 = (a^2 - 2)^2 - 4 = a^4 - 4a^2$, $A^2 = a^2 - 4$, and thus $a = \sqrt{A^2 + 4}$, as $a \neq 0$ nor $-\sqrt{A^2 + 4}$ because $a > 2$. **A1 E1**

So as the result is true for y_1 , and we have shown it to be true for y_{k+1} if it is true for y_k , it is true by induction for all positive integer n that

$$y_n = \frac{\sqrt{{x_{n+1}}^2 - 4}}{x_{n+1}}$$

E1 (8)

As $a > 2$ from (ii) $x_n \to \infty$ as $n \to \infty$ **M1** and thus using result just proved, $y_n \to 1$ as $n \to \infty$, i.e. the sequence converges. ***A1 (2)**

9.

Using the sine rule, from triangle PQR

$$\frac{PR}{\sin\theta} = \frac{PQ}{\sin\left(\frac{2\pi}{3} - \varphi\right)}$$

M1 A1

From triangle PQC

$$\frac{PQ}{\sin\frac{\pi}{3}} = \frac{a - x}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

A1

From triangle PBR

$$\frac{PR}{\sin\frac{\pi}{3}} = \frac{x}{\sin\varphi}$$

A1

Eliminating PR and PQ between these three equations

$$x\sin\frac{\pi}{3}\sin\left(\frac{2\pi}{3} - \varphi\right)\sin\left(\frac{2\pi}{3} - \theta\right) = \sin\varphi\ \sin\theta\,(a - x)\sin\frac{\pi}{3}$$

M1 A1

Hence

$$x\left(\frac{\sqrt{3}}{2}\cos\varphi + \frac{1}{2}\sin\varphi\right)\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right) = (a - x)\sin\varphi\ \sin\theta$$

giving

$$\left(\sqrt{3}\ \cot\varphi + 1\right)\left(\sqrt{3}\ \cot\theta + 1\right)x = 4(a - x)$$

as required. **M1 *A1 (8)**

If the ball has speed v_1 moving from P to Q, speed v_2 moving from Q to R, and speed v_3 moving from R to P,

then CLM at Q parallel to CA gives $v_1\cos\left(\frac{2\pi}{3} - \theta\right) = v_2\cos\frac{\pi}{3}$ and NELI perpendicular to CA gives $ev_1\sin\left(\frac{2\pi}{3} - \theta\right) = v_2\sin\frac{\pi}{3}$, and dividing these gives $e\tan\left(\frac{2\pi}{3} - \theta\right) = \tan\frac{\pi}{3}$

M1 A1

and similarly,

CLM at R parallel to AB gives $v_2\cos\frac{\pi}{3} = v_3\cos\varphi$ and NELI perpendicular to AB gives

$ev_2\sin\frac{\pi}{3} = v_3\sin\varphi$, and dividing these gives $e\tan\frac{\pi}{3} = \tan\varphi$. **A1**

$e\tan\left(\frac{2\pi}{3} - \theta\right) = \tan\frac{\pi}{3}$ yields $e\frac{-\sqrt{3}-\tan\theta}{1-\sqrt{3}\tan\theta} = \sqrt{3}$ **M1** which simplifies to

$e\left(\sqrt{3}+\tan\theta\right) = \sqrt{3}\left(\sqrt{3}\tan\theta - 1\right)$, or in turn, $(3-e)\tan\theta = \sqrt{3}(1+e)$ and so $\cot\theta = \frac{(3-e)}{\sqrt{3}(1+e)}$ **A1**

$e\tan\frac{\pi}{3} = \tan\varphi$ yields $\cot\varphi = \frac{1}{e\sqrt{3}}$ **A1**

Substituting these two expressions into the first result of the question,

$$\left(\frac{1}{e}+1\right)\left(\frac{(3-e)}{(1+e)}+1\right)x = 4(a-x)$$

M1

This simplifies to

$$x\,\frac{1+e}{e}\,\frac{4}{1+e} = 4(a-x)$$

that is

$$x = e(a-x)$$

so

$$x = \frac{ae}{1+e}$$

as required. ***A1 (8)**

To continue the motion at P, then similarly to before, the third impact gives $e\tan\left(\frac{2\pi}{3}-\varphi\right) = \tan\theta$

M1

So

$$\tan\theta = e\frac{-\sqrt{3}-\tan\varphi}{1-\sqrt{3}\tan\varphi} = e\frac{\sqrt{3}(e+1)}{3e-1}$$

and thus, using the previously found result for $\cot\theta$

$$\frac{(3-e)}{\sqrt{3}(1+e)} = \frac{3e-1}{\sqrt{3}(e+1)e}$$

M1 A1

That is $e(3-e) = 3e-1$, that is $e^2 = 1$ and as $e \geq 0$, $e = 1$ (and not -1) ***B1 (4)**

10. (i) At time t, the point where the string is tangential to the cylinder, **M1** say T is at $(a\cos\theta, a\sin\theta)$, **A1** the piece of string that remains straight is of length $b - a\theta$, **M1**, the vector representing the string is thus $(b - a\theta)\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$ **dM1 A1** so the particle is at the point $(a\cos\theta - (b - a\theta)\sin\theta, a\sin\theta + (b - a\theta)\cos\theta)$. **M1 A1 (7)**

$$\dot{x} = -a\dot{\theta}\sin\theta - (b - a\theta)\dot{\theta}\cos\theta + a\dot{\theta}\sin\theta = -(b - a\theta)\dot{\theta}\cos\theta$$

$$\dot{y} = a\dot{\theta}\cos\theta - (b - a\theta)\dot{\theta}\sin\theta - a\dot{\theta}\cos\theta = -(b - a\theta)\dot{\theta}\sin\theta$$

M1 A1

Thus the speed is $\sqrt{\left((b - a\theta)\dot{\theta}\cos\theta\right)^2 + \left((b - a\theta)\dot{\theta}\sin\theta\right)^2} = (b - a\theta)\dot{\theta}$ as required. **M1 A1 (4)**

(ii) The only horizontal force on the particle is the tension in the string, which is perpendicular to the velocity at any time, so kinetic energy is conserved. **E1** Therefore,

$$\frac{1}{2}m\left((b - a\theta)\dot{\theta}\right)^2 = \frac{1}{2}mu^2$$

M1

and so, as $(b - a\theta)\dot{\theta}$ and u are both positive $(b - a\theta)\dot{\theta} = u$ ***A1 (3)**

(iii) The tension in the string, using instantaneous circular motion, at time t is

$$\frac{mu^2}{(b - a\theta)}$$

M1 A1

As $(b - a\theta)\dot{\theta} = u$, integrating with respect to t,

$$b\theta - \frac{a\theta^2}{2} = ut + c$$

M1

but when $t = 0$, $\theta = 0$ so $c = 0$. **M1 A1**

Thus, $b\theta - \frac{a\theta^2}{2} = ut$

i.e.

$$\theta^2 - \frac{2b\theta}{a} + \frac{b^2}{a^2} = \frac{b^2}{a^2} - \frac{2ut}{a} = \frac{b^2 - 2aut}{a^2}$$

Alternatively, integrating $(b - a\theta)\dot{\theta} = u$ with respect to t,

$$-\frac{(b - a\theta)^2}{2a} = ut + k$$

M1

When $t = 0$, $\theta = 0$ so $k = -\frac{b^2}{2a}$ **M1 A1**

$$\frac{(b - a\theta)^2}{2a} = \frac{b^2}{2a} - ut = \frac{b^2 - 2aut}{2a}$$

Thus, taking positive roots,

$$\frac{b - a\theta}{a} = \frac{\sqrt{b^2 - 2aut}}{a}$$

Hence, the tension is

$$\frac{mu^2}{\sqrt{b^2 - 2aut}}$$

***A1 (6)**

11. (i)

$$P(Y=n) = P(n \le X < n+1) = \int_n^{n+1} \lambda\, e^{-\lambda x}\, dx = \left[-\, e^{-\lambda x}\right]_n^{n+1} = -\, e^{-\lambda(n+1)} + \, e^{-\lambda n}$$

M1 **M1**

$$= \left(1 - e^{-\lambda}\right) e^{-\lambda n}$$

as required. ***A1 (3)**

(ii)

$$P(Z < z) = \sum_{r=0}^{\infty} P(r \le X < r+z) = \sum_{r=0}^{\infty} \int_r^{r+z} \lambda\, e^{-\lambda x}\, dx = \sum_{r=0}^{\infty} \left[-\, e^{-\lambda x}\right]_r^{r+z}$$

M1 **M1**

$$= \sum_{r=0}^{\infty} \left(-\, e^{-\lambda(r+x)} + \, e^{-\lambda r}\right) = \sum_{r=0}^{\infty} \left(1 - e^{-\lambda z}\right) e^{-\lambda r}$$

M1 A1

$$= \left(1 - e^{-\lambda z}\right) \frac{1}{1 - e^{-\lambda}} = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}$$

using sum of an infinite GP with magnitude of common ratio less than one.

M1 *A1 (6)

(iii) As $P(Z < z) = \frac{1-e^{-\lambda z}}{1-e^{-\lambda}}$, $f_Z(z) = \frac{d}{dz}\left(\frac{1-e^{-\lambda z}}{1-e^{-\lambda}}\right) = \frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}}$ **M1**

so

$$E(Z) = \int_0^1 z\, \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda}}\, dz = \frac{1}{1 - e^{-\lambda}} \left\{ \left[-z e^{-\lambda z}\right]_0^1 + \int_0^1 e^{-\lambda z}\, dz \right\}$$

M1 **M1**

$$= \frac{1}{1 - e^{-\lambda}} \left\{ -e^{-\lambda} - \left[\frac{e^{-\lambda z}}{\lambda}\right]_0^1 \right\} = \frac{1}{1 - e^{-\lambda}} \left\{ -e^{-\lambda} - \frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda} \right\}$$

A1

$$= \frac{1}{\lambda} - \frac{e^{-\lambda}}{1 - e^{-\lambda}}$$

or alternatively

$$\frac{1}{\lambda}\, \frac{\left(1 - (\lambda + 1) e^{-\lambda}\right)}{1 - e^{-\lambda}}$$

A1 (5)

(iv)

$$P(Y = n \; and \; z_1 < Z < z_2) = P(n + z_1 < X < n + z_2)$$

$$= \int_{n+z_1}^{n+z_2} \lambda \, e^{-\lambda x} \, dx = \left[- \, e^{-\lambda x}\right]_{n+z_1}^{n+z_2} = - \, e^{-\lambda(n+z_2)} + \, e^{-\lambda(n+z_1)} = e^{-\lambda n}\left(e^{-\lambda z_1} - e^{-\lambda z_2}\right)$$

M1 **A1**

$$P(Y = n \; and \; z_1 < Z < z_2) = e^{-\lambda n}\left(e^{-\lambda z_1} - e^{-\lambda z_2}\right)$$

$$= \left(1 - e^{-\lambda}\right)e^{-\lambda n}\left(\frac{1 - e^{-\lambda z_2}}{1 - e^{-\lambda}} - \frac{1 - e^{-\lambda z_1}}{1 - e^{-\lambda}}\right)$$

M1 A1

$$= P(Y = n) \times P(\, z_1 < Z < z_2)$$ **M1**

so Y and Z are independent. **E1 (6)**

12. (i)

$$P(X_{12}=1)=\frac{1}{6}\,,P(X_{12}=0)=\frac{5}{6}\,,P(X_{23}=1)=\frac{1}{6}\,,P(X_{23}=0)=\frac{5}{6}$$

If $X_{23}=1$, then players 2 and 3 score the same as one another. In that case, $X_{12}=1$ would mean that player 1 also obtained that same score so $P(X_{12}=1|X_{23}=1)=\frac{1}{6}=P(X_{12}=1)$.

If $X_{23}=1$, $X_{12}=0$ would mean that player 1 obtained a different score so

$$P(X_{12}=0|X_{23}=1)=\frac{5}{6}=P(X_{12}=0)$$

If $X_{23}=0$, then players 2 and 3 score differently to one another. In that case, $X_{12}=1$ would mean that player 1 also obtained the same score as player 2 so $P(X_{12}=1|X_{23}=0)=\frac{1}{6}=P(X_{12}=1)$

If $X_{23}=0$, $X_{12}=0$ would mean that player 1 obtained a different score to player 2 so

$$P(X_{12}=0|X_{23}=0)=\frac{5}{6}=P(X_{12}=0)$$

Hence X_{12} is independent of X_{23}. **M1 A1 (2)**

Alternatively,

X_{12} X_{23}

1 1 requires players 2 and 3 to both score same as player 1 so

$$P(X_{12}=1\ and\ X_{23}=1\,)=\frac{1}{36}=\frac{1}{6}\times\frac{1}{6}=P(X_{12}=1)\times P(X_{23}=1)$$

1 0 requires player 2 to score the same as player as player 1, and player 3 score differently so

$$P(X_{12}=1\ and\ X_{23}=0\,)=\frac{5}{36}=\frac{1}{6}\times\frac{5}{6}=P(X_{12}=1)\times P(X_{23}=0)$$

0 1 requires players 2 and 3 to score the same as one another, and player 1 score differently so

$$P(X_{12}=0\ and\ X_{23}=1\,)=\frac{5}{36}=\frac{5}{6}\times\frac{1}{6}=P(X_{12}=0)\times P(X_{23}=1)$$

0 0 requires both player 1 and 3 to score differently to player 2 so

$$P(X_{12}=0\ and\ X_{23}=0\,)=\frac{25}{36}=\frac{5}{6}\times\frac{5}{6}=P(X_{12}=0)\times P(X_{23}=0)$$

Hence X_{12} is independent of X_{23}. **M1 A1 (2)**

If total score is T , then

$$T=\sum_{i<j}X_{ij}$$

M1

so

$$E(T) = E\left(\sum_{i<j} X_{ij}\right) = \sum_{i<j} E(X_{ij}) = {}^{n}C_2\ E(X_{12}) = {}^{n}C_2\ \left(1 \times \frac{1}{6} + 0 \times \frac{5}{6}\right) = \frac{n(n-1)}{12}$$

M1 **A1**

$$Var(T) = Var\left(\sum_{i<j} X_{ij}\right) = \sum_{i<j} Var(X_{ij}) = {}^{n}C_2\, Var(X_{12}) = {}^{n}C_2\ \left(1^2 \times \frac{1}{6} + 0^2 \times \frac{5}{6} - \frac{1}{6}^2\right)$$

M1

$$= \frac{5n(n-1)}{72}$$

A1 (5)

(ii)

$$Var(Y_1 + Y_2 + \ \dots\ + Y_m) = E((Y_1 + Y_2 + \ \dots\ + Y_m)^2) - [E(Y_1 + Y_2 + \ \dots\ + Y_m)]^2$$

$$= E\left({Y_1}^2 + {Y_2}^2 + \ \dots + {Y_m}^2 + 2Y_1Y_2 + 2Y_1Y_3 + \ \dots + 2Y_{n-1}Y_n\right) - [E(Y_1) + E(Y_2) + \ \dots\ + E(Y_m)]^2$$

$$= E\left(\sum_{i=1}^{m} {Y_i}^2\right) + 2E\left(\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} Y_iY_j\right) - (0 + 0 + \dots + 0)^2$$

$$= \sum_{i=1}^{m} E({Y_i}^2) + 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} E(Y_iY_j)$$

M1 *A1 (2)

(iii)

$$P(Z_{12} = 1) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

If $Z_{23} = 1$ then player 2 has rolled an even score and player 3 has scored the same so, in this case, for $Z_{12} = 1$, require player 1 to roll the score that player has so $P(Z_{12} = 1 | Z_{23} = 1) = \frac{1}{6}$.

Therefore, $P(Z_{12} = 1) \neq P(Z_{12} = 1 | Z_{23} = 1)$ and thus Z_{12} and Z_{23} are not independent.

Alternatively,
$P(Z_{12} = 1) = \frac{1}{12}$, $P(Z_{23} = 1) = \frac{1}{12}$

For $Z_{12} = 1$ and $Z_{23} = 1$ we require all three players to score the same even number so

$$P(Z_{12} = 1\ and\ Z_{23} = 1\ ) = \frac{3}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{72} \neq \frac{1}{12} \times \frac{1}{12} = P(Z_{12} = 1) \times P(Z_{23} = 1)$$

and thus they are not independent. **M1 A1 (2)**

Using part (ii), let $Y_1 = Z_{12}$, let $Y_2 = Z_{13}$, … let $Y_m = Z_{(n-1)n}$

(and with $m = {}^{n}C_2 = \frac{n(n-1)}{2}$).

$P(Z_{12}=1)=\frac{1}{12}$, $P(Z_{12}=-1)=\frac{1}{12}$, $P(Z_{12}=0)=\frac{5}{6}$ so $E(Z_{12})=0$ and $E\left({Z_{12}}^2\right)=\frac{1}{6}$ and likewise for all other Z (Y!). **B1** **B1**

If total score is U , then

$$U=\sum_{i<j} Z_{ij}$$

so

$$E(U)=E\left(\sum_{i<j} Z_{ij}\right)=\sum_{i<j} E\left(Z_{ij}\right)=0$$

B1

which means we can apply the result of (ii).

If $Z_{12}=1$ then $Z_{13}=1$ or $Z_{13}=0$

If $Z_{12}=-1$ then $Z_{13}=-1$ or $Z_{13}=0$

Otherwise $Z_{12}=0$

So $E(Z_{12}Z_{13})=1\times 1\times\frac{1}{72}+\ -1\times -1\times\frac{1}{72}=\frac{1}{36}$ **M1 A1**

So

$$Var(U)=\frac{n(n-1)}{2}\times\frac{1}{6}+2\ \times n\times\ {}^{n-1}C_2\ \times\ \frac{1}{36}=\frac{n(n-1)}{12}+\frac{n(n-1)(n-2)}{36}$$

M1 **M1 A1**

$$=\frac{n(n-1)}{36}\big(3+(n-2)\big)=\frac{n(n-1)(n+1)}{36}=\frac{n(n^2-1)}{36}$$

***A1 (9)**