STEP Support Programme

STEP 2 Calculus Questions

1 **2005 S2 Q1**

Find the three values of x for which the derivative of $x^2 \mathrm{e}^{-x^2}$ is zero.

Given that a and b are distinct positive numbers, find a polynomial $\mathrm{P}(x)$ such that the derivative of $\mathrm{P}(x)\mathrm{e}^{-x^2}$ is zero for $x = 0$, $x = \pm a$ and $x = \pm b$, but for no other values of x.

2 **2006 S2 Q4**

By making the substitution $x = \pi - t$, show that

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \tfrac{1}{2}\pi \int_0^{\pi} \mathrm{f}(\sin x)\,\mathrm{d}x\,,$$

where $\mathrm{f}(\sin x)$ is a given function of $\sin x$.

Evaluate the following integrals:

(i) $\displaystyle\int_0^{\pi} \frac{x \sin x}{3 + \sin^2 x}\,\mathrm{d}x\,;$

(ii) $\displaystyle\int_0^{2\pi} \frac{x \sin x}{3 + \sin^2 x}\,\mathrm{d}x\,;$

(iii) $\displaystyle\int_0^{\pi} \frac{x\left|\sin 2x\right|}{3 + \sin^2 x}\,\mathrm{d}x\,.$

3 **2007 S2 Q6**

(i) Differentiate $\ln\left(x + \sqrt{3 + x^2}\,\right)$ and $x\sqrt{3 + x^2}$ and simplify your answers.

Hence find $\int \sqrt{3 + x^2}\,\mathrm{d}x$.

(ii) Find the two solutions of the differential equation

$$3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

that satisfy $y = 0$ when $x = 1$.

4 2008 S2 Q7

(i) By writing $y = u(1+x^2)^{\frac{1}{2}}$, where u is a function of x, find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = xy + \frac{x}{1+x^2}$$

for which $y = 1$ when $x = 0$.

(ii) Find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y + \frac{x^2}{1+x^3}$$

for which $y = 1$ when $x = 0$.

(iii) Give, without proof, a conjecture for the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^{n-1}y + \frac{x^{n-1}}{1+x^n}$$

for which $y = 1$ when $x = 0$, where n is an integer greater than 1.

If you are finding question **4** quite tricky you might like to tackle the question below first (which uses the same ideas but is a STEP I question).

5 2010 S1 Q6

Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0\,. \qquad (*)$$

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

By setting $\frac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.