

## STEP Support Programme

### STEP 2 Equations and Inequalities Questions

#### 1 1994 S2 Q5

(i) Show that the equation

$$(x-1)^4 + (x+1)^4 = c$$

has exactly two real roots if  $c > 2$ , one root if  $c = 2$  and no roots if  $c < 2$ .

(ii) How many real roots does the equation  $(x-3)^4 + (x-1)^4 = c$  have?

(iii) How many real roots does the equation  $|x-3| + |x-1| = c$  have?

(iv) How many real roots does the equation  $(x-3)^3 + (x-1)^3 = c$  have?

[The answers to parts (ii), (iii) and (iv) may depend on the value of  $c$ . You should give reasons for your answers.]

#### 2 2006 S2 Q2

Using the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

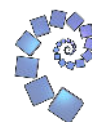
show that  $e > \frac{8}{3}$ .

Show that  $n! > 2^n$  for  $n \geq 4$  and hence show that  $e < \frac{67}{24}$ .

Show that the curve with equation

$$y = 3e^{2x} + 14 \ln\left(\frac{4}{3} - x\right), \quad x < \frac{4}{3}$$

has a minimum turning point between  $x = \frac{1}{2}$  and  $x = 1$  and give a sketch to show the shape of the curve.



### 3 2003 S2 Q1

Consider the equations

$$\begin{aligned} ax - y - z &= 3, \\ 2ax - y - 3z &= 7, \\ 3ax - y - 5z &= b, \end{aligned}$$

where  $a$  and  $b$  are given constants.

- (i) In the case  $a = 0$ , show that the equations have a solution if and only if  $b = 11$ .
- (ii) In the case  $a \neq 0$  and  $b = 11$  show that the equations have a solution with  $z = \lambda$  for any given number  $\lambda$ .
- (iii) In the case  $a = 2$  and  $b = 11$  find the solution for which  $x^2 + y^2 + z^2$  is least.
- (iv) Find a value for  $a$  for which there is a solution such that  $x > 10^6$  and  $y^2 + z^2 < 1$ .

### 4 2010 S2 Q7

- (i) By considering the positions of its turning points, show that the curve with equation

$$y = x^3 - 3qx - q(1 + q),$$

where  $q > 0$  and  $q \neq 1$ , crosses the  $x$ -axis once only.

- (ii) Given that  $x$  satisfies the cubic equation

$$x^3 - 3qx - q(1 + q) = 0,$$

and that

$$x = u + q/u,$$

obtain a quadratic equation satisfied by  $u^3$ . Hence find the real root of the cubic equation in the case  $q > 0$ ,  $q \neq 1$ .

- (iii) The quadratic equation

$$t^2 - pt + q = 0$$

has roots  $\alpha$  and  $\beta$ . Show that

$$\alpha^3 + \beta^3 = p^3 - 3qp.$$

It is given that one of these roots is the square of the other. By considering the expression  $(\alpha^2 - \beta)(\beta^2 - \alpha)$ , find a relationship between  $p$  and  $q$ . Given further that  $q > 0$ ,  $q \neq 1$  and  $p$  is real, determine the value of  $p$  in terms of  $q$ .

