STEP Support Programme

STEP 2 Equations and Inequalities Questions

1 **1994 S2 Q5**

(i) Show that the equation

$$(x-1)^4 + (x+1)^4 = c$$

has exactly two real roots if $c > 2$, one root if $c = 2$ and no roots if $c < 2$.

(ii) How many real roots does the equation $(x-3)^4 + (x-1)^4 = c$ have?

(iii) How many real roots does the equation $|x-3| + |x-1| = c$ have?

(iv) How many real roots does the equation $(x-3)^3 + (x-1)^3 = c$ have?

[The answers to parts **(ii)**, **(iii)** and **(iv)** may depend on the value of c. You should give reasons for your answers.]

2 **2006 S2 Q2**

Using the series

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

show that $\mathrm{e} > \frac{8}{3}$.

Show that $n! > 2^n$ for $n \geqslant 4$ and hence show that $\mathrm{e} < \frac{67}{24}$.

Show that the curve with equation

$$y = 3\mathrm{e}^{2x} + 14\ln(\tfrac{4}{3} - x)\,, \qquad x < \tfrac{4}{3}$$

has a minimum turning point between $x = \frac{1}{2}$ and $x = 1$ and give a sketch to show the shape of the curve.

3 2003 S2 Q1

Consider the equations

$$\begin{aligned} ax - y - \ z &= 3\,, \\ 2ax - y - 3z &= 7\,, \\ 3ax - y - 5z &= b\,, \end{aligned}$$

where a and b are given constants.

(i) In the case $a = 0$, show that the equations have a solution if and only if $b = 11$.

(ii) In the case $a \neq 0$ and $b = 11$ show that the equations have a solution with $z = \lambda$ for any given number λ.

(iii) In the case $a = 2$ and $b = 11$ find the solution for which $x^2 + y^2 + z^2$ is least.

(iv) Find a value for a for which there is a solution such that $x > 10^6$ and $y^2 + z^2 < 1$.

4 2010 S2 Q7

(i) By considering the positions of its turning points, show that the curve with equation

$$y = x^3 - 3qx - q(1 + q)\,,$$

where $q > 0$ and $q \neq 1$, crosses the x-axis once only.

(ii) Given that x satisfies the cubic equation

$$x^3 - 3qx - q(1 + q) = 0\,,$$

and that

$$x = u + q/u\,,$$

obtain a quadratic equation satisfied by u^3. Hence find the real root of the cubic equation in the case $q > 0$, $q \neq 1$.

(iii) The quadratic equation

$$t^2 - pt + q = 0$$

has roots α and β. Show that

$$\alpha^3 + \beta^3 = p^3 - 3qp\,.$$

It is given that one of these roots is the square of the other. By considering the expression $(\alpha^2 - \beta)(\beta^2 - \alpha)$, find a relationship between p and q. Given further that $q > 0$, $q \neq 1$ and p is real, determine the value of p in terms of q.