

STEP Support Programme

STEP 3 Vectors Questions

1 SPECIMEN S2 Q9

- (i) Let \mathbf{a} and \mathbf{b} be given vectors with $\mathbf{b} \neq \mathbf{0}$, and let \mathbf{x} be a position vector. Find the condition for the sphere $|\mathbf{x}| = R$, where $R > 0$, and the plane $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ to intersect.

When this condition is satisfied, find the radius and the position vector of the centre of the circle in which the plane and sphere intersect.

- (ii) Let \mathbf{c} be a given vector, with $\mathbf{c} \neq \mathbf{0}$. The vector \mathbf{x}' is related to the vector \mathbf{x} by

$$\mathbf{x}' = \mathbf{x} - \frac{2(\mathbf{x} \cdot \mathbf{c})\mathbf{c}}{|\mathbf{c}|^2}.$$

Interpret this relation geometrically.

2 92 S2 Q9

Let \mathbf{a}, \mathbf{b} and \mathbf{c} be the position vectors of points A, B and C in three-dimensional space. Suppose that A, B, C and the origin O are not all in the same plane. Describe the locus of the point whose position vector \mathbf{r} is given by

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where λ and μ are scalar parameters. By writing this equation in the form $\mathbf{r} \cdot \mathbf{n} = p$ for a suitable vector \mathbf{n} and scalar p , show that

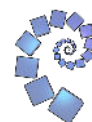
$$-(\lambda + \mu)\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \lambda\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) + \mu\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

for all scalars λ, μ .

Deduce that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

Say briefly what happens if A, B, C and O are all in the same plane.



3 93 S2 Q4

Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\mathbf{m}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\mathbf{m}_2$ respectively, where \mathbf{m}_1 and \mathbf{m}_2 are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{m}_1 \times \mathbf{m}_2)|}{|\mathbf{m}_1 \times \mathbf{m}_2|}.$$

(i) Find the shortest distance between the lines in the case

$$\mathbf{p}_1 = (2, 1, -1) \quad \mathbf{p}_2 = (1, 0, -2) \quad \mathbf{m}_1 = \frac{1}{5}(4, 3, 0) \quad \mathbf{m}_2 = \frac{1}{\sqrt{10}}(0, -3, 1).$$

(ii) Two aircraft, A_1 and A_2 , are flying in the directions given by the unit vectors \mathbf{m}_1 and \mathbf{m}_2 at constant speeds v_1 and v_2 . At time $t = 0$ they pass the points \mathbf{p}_1 and \mathbf{p}_2 , respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2 |v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - [(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1 . How close is that, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}_1$ and \mathbf{m}_2 are as in (i)?

4 95 S3 Q8

A plane π in 3-dimensional space is given by the vector equation $\mathbf{r} \cdot \mathbf{n} = p$, where \mathbf{n} is a unit vector and p is a non-negative real number. If \mathbf{x} is the position vector of a general point X , find the equation of the normal to π through X and the perpendicular distance of X from π .

The unit circles C_i , $i = 1, 2$, with centres \mathbf{r}_i , lie in the planes π_i given by $\mathbf{r} \cdot \mathbf{n}_i = p_i$, where the \mathbf{n}_i are unit vectors, and p_i are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number λ such that

$$\mathbf{r}_1 + \lambda\mathbf{n}_1 = \mathbf{r}_2 \pm \lambda\mathbf{n}_2.$$

Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0$$

and that

$$(p_1 - \mathbf{n}_1 \cdot \mathbf{r}_2)^2 = (p_2 - \mathbf{n}_2 \cdot \mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.

