

STEP Support Programme

STEP 3 Calculus Questions

1 2003 S3 Q1

Given that $x + a > 0$ and $x + b > 0$, and that $b > a$, show that

$$\frac{d}{dx} \arcsin \left(\frac{x+a}{x+b} \right) = \frac{\sqrt{b-a}}{(x+b) \sqrt{a+b+2x}}$$

and find $\frac{d}{dx} \operatorname{arcosh} \left(\frac{x+b}{x+a} \right)$.

Hence, or otherwise, integrate, for $x > -1$,

(i) $\int \frac{1}{(x+1)\sqrt{x+3}} dx$,

(ii) $\int \frac{1}{(x+3)\sqrt{x+1}} dx$.

[You may use the results $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx} \operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$.]

2 2010 S3 Q2

In this question, a is a positive constant.

(i) Express $\cosh a$ in terms of exponentials.

By using partial fractions, prove that

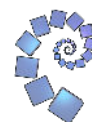
$$\int_0^1 \frac{1}{x^2 + 2x \cosh a + 1} dx = \frac{a}{2 \sinh a}.$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x \sinh a - 1} dx$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2 \cosh a + 1} dx.$$



3 2002 S3 Q1

Find the area of the region between the curve $y = \frac{\ln x}{x}$ and the x -axis, for $1 \leq x \leq a$. What happens to this area as a tends to infinity?

Find the volume of the solid obtained when the region between the curve $y = \frac{\ln x}{x}$ and the x -axis, for $1 \leq x \leq a$, is rotated through 2π radians about the x -axis. What happens to this volume as a tends to infinity?

4 2004 S3 Q7

For $n = 1, 2, 3, \dots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n} dt.$$

By considering the greatest value taken by $\frac{t}{t+1}$ for $0 \leq t \leq 1$ show that $I_{n+1} < \frac{1}{2}I_n$.

Show also that $I_{n+1} = -\frac{1}{n 2^n} + I_n$.

Deduce that $I_n < \frac{1}{n 2^{n-1}}$.

Prove that

$$\ln 2 = \sum_{r=1}^n \frac{1}{r 2^r} + I_{n+1}$$

and hence show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.

