

## STEP Support Programme

### STEP 3 Complex Numbers Questions

#### 1 2010 S3 Q3

For any given positive integer  $n$ , a number  $a$  (which may be complex) is said to be a *primitive  $n$ th root of unity* if  $a^n = 1$  and there is no integer  $m$  such that  $0 < m < n$  and  $a^m = 1$ . Write down the two primitive 4th roots of unity.

Let  $C_n(x)$  be the polynomial such that the roots of the equation  $C_n(x) = 0$  are the primitive  $n$ th roots of unity, the coefficient of the highest power of  $x$  is one and the equation has no repeated roots. Show that  $C_4(x) = x^2 + 1$ .

- (i) Find  $C_1(x)$ ,  $C_2(x)$ ,  $C_3(x)$ ,  $C_5(x)$  and  $C_6(x)$ , giving your answers as unfactorised polynomials.
- (ii) Find the value of  $n$  for which  $C_n(x) = x^4 + 1$ .
- (iii) Given that  $p$  is prime, find an expression for  $C_p(x)$ , giving your answer as an unfactorised polynomial.
- (iv) Prove that there are no positive integers  $q$ ,  $r$  and  $s$  such that  $C_q(x) \equiv C_r(x)C_s(x)$ .

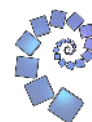
#### 2 2007 S3 Q6

The distinct points  $P$ ,  $Q$ ,  $R$  and  $S$  in the Argand diagram lie on a circle of radius  $a$  centred at the origin and are represented by the complex numbers  $p$ ,  $q$ ,  $r$  and  $s$ , respectively. Show that

$$pq = -a^2 \frac{p - q}{p^* - q^*}.$$

Deduce that, if the chords  $PQ$  and  $RS$  are perpendicular, then  $pq + rs = 0$ .

The distinct points  $A_1, A_2, \dots, A_n$  (where  $n \geq 3$ ) lie on a circle. The points  $B_1, B_2, \dots, B_n$  lie on the same circle and are chosen so that the chords  $B_1B_2, B_2B_3, \dots, B_nB_1$  are perpendicular, respectively, to the chords  $A_1A_2, A_2A_3, \dots, A_nA_1$ . Show that, for  $n = 3$ , there are only two choices of  $B_1$  for which this is possible. What is the corresponding result for  $n = 4$ ? State the corresponding results for values of  $n$  greater than 4.



### 3 2013 S3 Q4

Show that  $(z - e^{i\theta})(z - e^{-i\theta}) = z^2 - 2z \cos \theta + 1$ .

Write down the  $(2n)$ th roots of  $-1$  in the form  $e^{i\theta}$ , where  $-\pi < \theta \leq \pi$ , and deduce that

$$z^{2n} + 1 = \prod_{k=1}^n \left( z^2 - 2z \cos \left( \frac{(2k-1)\pi}{2n} \right) + 1 \right).$$

Here,  $n$  is a positive integer, and the  $\prod$  notation denotes the product.

(i) By substituting  $z = i$  show that, when  $n$  is even,

$$\cos \left( \frac{\pi}{2n} \right) \cos \left( \frac{3\pi}{2n} \right) \cos \left( \frac{5\pi}{2n} \right) \cdots \cos \left( \frac{(2n-1)\pi}{2n} \right) = (-1)^{\frac{1}{2}n} 2^{1-n}.$$

(ii) Show that, when  $n$  is odd,

$$\cos^2 \left( \frac{\pi}{2n} \right) \cos^2 \left( \frac{3\pi}{2n} \right) \cos^2 \left( \frac{5\pi}{2n} \right) \cdots \cos^2 \left( \frac{(n-2)\pi}{2n} \right) = n 2^{1-n}.$$

You may use without proof the fact that  $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$  when  $n$  is odd.

### 4 2006 S3 Q5

Show that the distinct complex numbers  $\alpha$ ,  $\beta$  and  $\gamma$  represent the vertices of an equilateral triangle (in clockwise or anti-clockwise order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.$$

Show that the roots of the equation

$$z^3 + az^2 + bz + c = 0 \tag{*}$$

represent the vertices of an equilateral triangle if and only if  $a^2 = 3b$ .

Under the transformation  $z = pw + q$ , where  $p$  and  $q$  are given complex numbers with  $p \neq 0$ , the equation (\*) becomes

$$w^3 + Aw^2 + Bw + C = 0. \tag{**}$$

Show that if the roots of equation (\*) represent the vertices of an equilateral triangle, then the roots of equation (\*\*) also represent the vertices of an equilateral triangle.

