STEP Support Programme

STEP 3 Differential Equations Questions

1 2007 S3 Q8

(i) Find functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$ such that $u = x$ and $u = \mathrm{e}^{-x}$ both satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \mathrm{a}(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{b}(x)u = 0\,.$$

For these functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$, write down the general solution of the equation.

Show that the substitution $y = \dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \dfrac{1-\mathrm{e}^{-x}}{3(x+\mathrm{e}^{-x})}$.

(ii) Find the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.

2 2010 S3 Q7

Given that $y = \cos(m \arcsin x)$, for $|x| < 1$, prove that

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y = 0\,.$$

Obtain a similar equation relating $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and a similar equation relating $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}$, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Conjecture and prove a relation between $\dfrac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}$, $\dfrac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}$ and $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$.

Obtain the first three non-zero terms of the Maclaurin series for y. Show that, if m is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin\theta$ beginning

$$1 - \frac{m^2\sin^2\theta}{2!} + \frac{m^2(m^2-2^2)\sin^4\theta}{4!} - \cdots\,. \qquad (|\theta| < \tfrac{1}{2}\pi)$$

State the degree of the polynomial.

3 2010 S3 Q8

Given that $\mathrm{P}(x) = \mathrm{Q}(x)\mathrm{R}'(x) - \mathrm{Q}'(x)\mathrm{R}(x)$, write down an expression for

$$\int \frac{\mathrm{P}(x)}{\big(\mathrm{Q}(x)\big)^2}\,\mathrm{d}x\,.$$

(i) By choosing the function $\mathrm{R}(x)$ to be of the form $a + bx + cx^2$, find

$$\int \frac{5x^2-4x-3}{(1+2x+3x^2)^2}\,\mathrm{d}x\,.$$

Show that the choice of $\mathrm{R}(x)$ is not unique and, by comparing the two functions $\mathrm{R}(x)$ corresponding to two different values of a, explain how the different choices are related.

(ii) Find the general solution of

$$(1+\cos x + 2\sin x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\sin x - 2\cos x)y = 5 - 3\cos x + 4\sin x\,.$$

4 2012 S3 Q7

A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time t of the drug in the blood and the brain respectively are $y(t)$ and $z(t)$. These satisfy

$$\dot{y} = -2(y - z)\,, \qquad \dot{z} = -\dot{y} - 3z\,,$$

where the dot denotes differentiation with respect to t.

Obtain a second order differential equation for y and hence derive the solution

$$y = A\mathrm{e}^{-t} + B\mathrm{e}^{-6t}\,, \qquad z = \tfrac{1}{2}A\mathrm{e}^{-t} - 2B\mathrm{e}^{-6t}\,,$$

where A and B are arbitrary constants.

(i) Obtain the solution that satisfies $z(0) = 0$ and $y(0) = 5$. The quantity of the drug in the brain for this solution is denoted by $z_1(t)$.

(ii) Obtain the solution that satisfies $z(0) = z(1) = c$, where c is a given constant. The quantity of the drug in the brain for this solution is denoted by $z_2(t)$.

(iii) Show that for $0 \leqslant t \leqslant 1$,

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t - n)\,,$$

provided c takes a particular value that you should find.