

STEP Support Programme

STEP 3 Differential Equations Questions

1 2007 S3 Q8

- (i) Find functions $a(x)$ and $b(x)$ such that $u = x$ and $u = e^{-x}$ both satisfy the equation

$$\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)u = 0.$$

For these functions $a(x)$ and $b(x)$, write down the general solution of the equation.

Show that the substitution $y = \frac{1}{3u} \frac{du}{dx}$ transforms the equation

$$\frac{dy}{dx} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \quad (*)$$

into

$$\frac{d^2u}{dx^2} + \frac{x}{1+x} \frac{du}{dx} - \frac{1}{1+x}u = 0$$

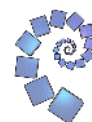
and hence show that the solution of equation (*) that satisfies $y = 0$ at $x = 0$ is given

$$\text{by } y = \frac{1 - e^{-x}}{3(x + e^{-x})}.$$

- (ii) Find the solution of the equation

$$\frac{dy}{dx} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.



2 2010 S3 Q7

Given that $y = \cos(m \arcsin x)$, for $|x| < 1$, prove that

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0.$$

Obtain a similar equation relating $\frac{d^3 y}{dx^3}$, $\frac{d^2 y}{dx^2}$ and $\frac{dy}{dx}$, and a similar equation relating $\frac{d^4 y}{dx^4}$, $\frac{d^3 y}{dx^3}$ and $\frac{d^2 y}{dx^2}$.

Conjecture and prove a relation between $\frac{d^{n+2} y}{dx^{n+2}}$, $\frac{d^{n+1} y}{dx^{n+1}}$ and $\frac{d^n y}{dx^n}$.

Obtain the first three non-zero terms of the Maclaurin series for y . Show that, if m is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin \theta$ beginning

$$1 - \frac{m^2 \sin^2 \theta}{2!} + \frac{m^2(m^2 - 2^2) \sin^4 \theta}{4!} - \dots \quad (|\theta| < \tfrac{1}{2}\pi)$$

State the degree of the polynomial.

3 2010 S3 Q8

Given that $P(x) = Q(x)R'(x) - Q'(x)R(x)$, write down an expression for

$$\int \frac{P(x)}{(Q(x))^2} dx.$$

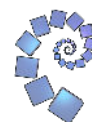
(i) By choosing the function $R(x)$ to be of the form $a + bx + cx^2$, find

$$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2} dx.$$

Show that the choice of $R(x)$ is not unique and, by comparing the two functions $R(x)$ corresponding to two different values of a , explain how the different choices are related.

(ii) Find the general solution of

$$(1 + \cos x + 2 \sin x) \frac{dy}{dx} + (\sin x - 2 \cos x)y = 5 - 3 \cos x + 4 \sin x.$$



4 2012 S3 Q7

A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time t of the drug in the blood and the brain respectively are $y(t)$ and $z(t)$. These satisfy

$$\dot{y} = -2(y - z), \quad \dot{z} = -\dot{y} - 3z,$$

where the dot denotes differentiation with respect to t .

Obtain a second order differential equation for y and hence derive the solution

$$y = Ae^{-t} + Be^{-6t}, \quad z = \frac{1}{2}Ae^{-t} - 2Be^{-6t},$$

where A and B are arbitrary constants.

- (i) Obtain the solution that satisfies $z(0) = 0$ and $y(0) = 5$. The quantity of the drug in the brain for this solution is denoted by $z_1(t)$.
- (ii) Obtain the solution that satisfies $z(0) = z(1) = c$, where c is a given constant. The quantity of the drug in the brain for this solution is denoted by $z_2(t)$.
- (iii) Show that for $0 \leq t \leq 1$,

$$z_2(t) = \sum_{n=-\infty}^0 z_1(t-n),$$

provided c takes a particular value that you should find.

