STEP Support Programme

Pure STEP 3 Questions

2012 S3 Q6

1 Preparation

The STEP question involves complex numbers and the Argand diagram. However, after you have put $z = x + \mathrm{i}y$ in the given equations and set the real and imaginary parts (separately) equal to zero, there are no more complex numbers left, and the Argand diagram is just the x-y plane.

In the first part, the locus given by the second equation (for which $y = 0$) is just the x-axis or part of it. Rather surprisingly, we are told to obtain p in terms of x, not x in terms of p. The question we are trying to answer is 'What values of x are allowed', in other words which values of x correspond to some value of p (it has to be a real value of p since that is what we were told and that is what we assumed when we took real and imaginary parts of the given equations). The equation

$$-p = x + \frac{1}{x}$$

shows that, for any value of x (except $x = 0$), there is a corresponding value of p. So the required locus is the whole of the x-axis except for $x = 0$.

It may be best to plunge straight into the STEP question without any preparation. But some of the sketches (especially for last part) are quite hard, so you might like to start with:

(i) Find the centre and radius of the circle $x^2 + y^2 + 2x = 0$.

(ii) For which values of x is $x^4 - x$ non-negative (i.e. greater than or equal to zero)?

For which values of x is $-x^2 - \dfrac{1}{x}$ non-negative?

(iii) Sketch the graphs $y = x - 1$, $y = \sqrt{x-1}$, and $y^2 = x - 1$, paying attention to the gradient of the second and third graphs at $x = 1$.

Note that in the second graph $y \geqslant 0$, but in the third y can be negative (though we must have $x - 1 \geqslant 0$).

2 The STEP 3 question

(i) Let $x + \mathrm{i}y$ be a root of the quadratic equation $z^2 + pz + 1 = 0$, where p is a real number. Show that $x^2 - y^2 + px + 1 = 0$ and $(2x + p)y = 0$. Show further that

$$\text{either} \quad p = -2x \quad \text{or} \quad p = -(x^2 + 1)/x \quad \text{with} \quad x \neq 0\,.$$

Hence show that the set of points in the Argand diagram that can (as p varies) represent roots of the quadratic equation consists of the real axis with one point missing and a circle. This set of points is called the *root locus* of the quadratic equation.

(ii) Obtain and sketch in the Argand diagram the root locus of the equation

$$pz^2 + z + 1 = 0\,.$$

(iii) Obtain and sketch in the Argand diagram the root locus of the equation

$$pz^2 + p^2 z + 2 = 0\,.$$

2001 S3 Q1

3 Preparation

(i) Use the chain rule to differentiate $\sin(\ln x)$. Use this to help you differentiate $e^{2\sin(\ln x)}$.

(ii) Use proof by induction to show that $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

You can show the base case $n = 1$ by first principles, or even just by sketching the line. It will help to note that $x^{(k+1)} = x \times x^k$.

(iii) Maclaurin's expansion is:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots,$$

(assuming this series converges).

Use this to find the first three non-zero terms in the expansion of $\sin x$. Hence estimate $\sin 0.1$ to seven decimal places, where 0.1 is in radians.

If 0.1 is in radians (not degrees), then we can use the result that the derivative of $\sin x$ is $\cos x$. You may well want to do the final calculation without a calculator, just because you can.

4 The STEP 3 question

Given that $y = \ln(x + \sqrt{x^2+1})$, show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2+1}}$.

Prove by induction that, for $n \geqslant 0$,

$$\left(x^2+1\right) y^{(n+2)} + (2n+1)\, xy^{(n+1)} + n^2 y^{(n)} = 0\,,$$

where $y^{(n)} = \dfrac{d^n y}{dx^n}$ and $y^{(0)} = y$.

Using this result in the case $x = 0$, or otherwise, show that the Maclaurin series for y begins

$$x - \frac{x^3}{6} + \frac{3x^5}{40}$$

and find the next non-zero term.

Discussion Note that the induction requires you to prove the result for $n \geqslant 0$, so make your 'base case' $n = 0$. Don't overlook the request tucked away at the end of the question (to find the next non-zero term)!

2009 S3 Q8

5 Preparation

(i) Although you probably won't study limits rigorously until you get to university, there are various properties of limits that you can understand and use without formal definitions.

For example, if $\lim_{x\to a} \mathrm{f}(x) = L$ and $\lim_{x\to a} \mathrm{g}(x) = K$ (where L and K are **finite**) then:

i. $\lim_{x\to a} c\mathrm{f}(x) = c\lim_{x\to a} \mathrm{f}(x) = cL$

ii. $\lim_{x\to a} \big(\mathrm{f}(x) + \mathrm{g}(x)\big) = \lim_{x\to a} \mathrm{f}(x) + \lim_{x\to a} \mathrm{g}(x) = L + K$

iii. $\lim_{x\to a} \big(\mathrm{f}(x)\mathrm{g}(x)\big) = \lim_{x\to a} \mathrm{f}(x) \times \lim_{x\to a} \mathrm{g}(x) = L \times K$

iv. $\lim_{x\to a} \dfrac{\mathrm{f}(x)}{\mathrm{g}(x)} = \dfrac{\lim_{x\to a} \mathrm{f}(x)}{\lim_{x\to a} \mathrm{g}(x)} = \dfrac{L}{K} \quad (K \neq 0).$

v. If $\mathrm{h}(x)$ is continuous, then $\lim_{x\to a} \mathrm{h}\big(\mathrm{g}(x)\big) = \mathrm{h}\big(\lim_{x\to a} \mathrm{g}(x)\big) = \mathrm{h}(K)\,.$

These all make good sense, and you would probably have assumed them to be true.

Evaluate the following limits, stating which of the above properties you have used:

(a) $\lim_{\theta\to 0} \dfrac{\sin\theta}{\cos\theta}$

(b) $\lim_{x\to\infty} 5(\mathrm{e}^{-x} + 1)$

(c) $\lim_{t\to\infty} \cos\frac{1}{t}$

(d) $\lim_{x\to\infty} \mathrm{e}^{\frac{1}{x}}$

(ii) By first writing $\mathrm{e}^{-t}t$ as $\dfrac{t}{\mathrm{e}^t}$ and then using the series expansion for e^t show that:

$$\lim_{t\to\infty} \mathrm{e}^{-t}t = 0\,.$$

There are various ways you can do this. You could, for example, start by dividing top and bottom by t. Note that you cannot use rule iii. above as the limit of t is not finite.

Show that $\lim_{t\to\infty} e^{-t}t^n = 0$ where n is a positive integer.

(iii) If $x = \mathrm{e}^{-t}$, how does x behave as $t \to \infty$? By setting substitution $x = \mathrm{e}^{-t}$ in the expression $x\ln x$, show that $\lim_{x\to 0}(x\ln x) = 0$.

(iv) Consider the integral $\displaystyle\int_0^1 \ln x \, \mathrm{d}x$. The function $\ln x$ is not defined when $x = 0$, but the integral can be defined as a limit, as follows.

(a) Evaluate $\displaystyle\int_r^1 \ln x \, \mathrm{d}x$ (see Assignment 24 if you are stuck).

(b) Now take the limit of this result as $r \to 0$.

6 The STEP 3 question

Let m be a positive integer and let n be a non-negative integer. Use the result $\displaystyle\lim_{t\to\infty} \mathrm{e}^{-mt}t^n = 0$ to show that

$$\lim_{x\to 0} x^m (\ln x)^n = 0\,.$$

(i) By writing x^x as $\mathrm{e}^{x \ln x}$ show that

$$\lim_{x\to 0} x^x = 1\,.$$

(ii) Let $I_n = \displaystyle\int_0^1 x^m (\ln x)^n \mathrm{d}x$. Show that

$$I_{n+1} = -\frac{n+1}{m+1} I_n$$

and hence evaluate I_n.

(iii) Show that

$$\int_0^1 x^x \mathrm{d}x = 1 - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^3 - \left(\tfrac{1}{4}\right)^4 + \cdots\,.$$

Discussion

Who would have thought the final integral would come out as such an interesting-looking series?

A small point: towards the end of part **(iii)** you have to integrate an infinite series, which you do by integrating each term and adding them all up. You have used the result that the integral of a sum is equal to the sum of the integrals, which certainly holds for the sum of a finite number of terms. However, it may not hold for an infinite number of terms; it depends on how well the series converges (as you will discover in your first university year). We are OK here, because the exponential series converges splendidly well, and you weren't intended to worry about it.

2006 S3 Q7

7 Preparation
Some remarks on hyperbolic functions

- Hyperbolic functions are defined in terms of exponentials (look it up if you don't know these definitions), in the same way as trigonometric functions could be defined in terms of complex exponentials ($\mathrm{e}^{\mathrm{i}\theta}$).
- All the formulae for trigonometric functions work similarly for hyperbolic functions provided you remember to change the sign in front of any product of two sinh functions. For example $\cosh^2\theta - \sinh^2\theta = 1$. This is not magic: it follows immediately from the identities $\sinh x = -\mathrm{i}\sin(\mathrm{i}x)$ and $\cosh x = \cos(\mathrm{i}x)$ which themselves follow directly from the definitions (try it if you haven't already done so). This "rule" is known as "Osbourn's rule", and first appeared in 1902 as "Mnemonic for hyperbolic functions" in the Mathematical Gazette.
- Although trigonometric and hyperbolic functions are algebraically related as above, they are very distinct geometrically. For example, $\cosh x \geqslant 1$ if x is real; and the equation $\sinh x = 0$ has only one real solution.

The hyperbolic functions $\sinh x$ (pronounced "sinch or shine") and $\cosh x$ are defined by:

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\,.$$

(i) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$ and find $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sinh x)$.

(ii) Sketch $\cosh x$ and $\sinh x$.

Note that when $x \to +\infty$, we can ignore the e^{-x} term.

(iii) Simplify $\cosh x + \sinh x$ and $\cosh x - \sinh x$.

(iv) Simplify $(\cosh x)^2 - (\sinh x)^2$.

(v) Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sinh x$, where $y = 2$ when $x = 0$.

(vi) Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \tanh y$, where $\tanh y = \dfrac{\sinh y}{\cosh y}$ and is pronounced "tanch" or "th-an". Write your answer in the form $\sinh y = \cdots$.

8 The STEP 3 question

(i) Solve the equation $u^2 + 2u\sinh x - 1 = 0$ giving u in terms of x.

Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

that satisfies $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ at $x = 0$.

(ii) Find the solution, not identically zero, of the differential equation

$$\sinh y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh y = 0$$

that satisfies $y = 0$ at $x = 0$, expressing your solution in the form $\cosh y = \mathrm{f}(x)$. Show that the asymptotes to the solution curve are $y = \pm(-x + \ln 4)$.

Discussion

Asymptotes are lines that the curve approaches as either x or y or both tend to infinity. As some terms get very big, other terms (such as constant terms) can be ignored. For example, as $x \to +\infty$, $\sqrt{x^2+1} \approx \sqrt{x^2} = x$ (since $x \to +\infty$ we have $x > 0$).

1988 S2 Q5

9 Preparation

(i) **(a)** Prove by induction that $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$.

(b) Find all solutions of the equation $z^5 = 1$, by writing z as $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r \geqslant 0$. You should justify each step carefully.

(c) Express $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(ii) If the roots of the equation $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, are α_1, α_2, $\ldots$, α_n, what is the expression for the product of the roots in terms of the coefficients a_i?

It just involves one coefficient, and depends on n. If you haven't seen this before, factorise the equation in the form $(x - \alpha_1)\cdots(x - \alpha_n) = 0$

10 The STEP 3 question

By considering the imaginary part of the equation $z^7 = 1$, or otherwise, find all the roots of the equation

$$t^6 - 21t^4 + 35t^2 - 7 = 0.$$

You should justify each step carefully.

Hence, or otherwise, prove that

$$\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7} = \sqrt{7}.$$

Find the corresponding result for

$$\tan\frac{2\pi}{n}\tan\frac{4\pi}{n}\cdots\tan\frac{(n-1)\pi}{n}$$

in the two cases $n = 9$ and $n = 11$.

This is a 30 year old STEP II question, included here as it involves roots of unity which are now on the STEP 3 specification (rather than STEP 2).

2013 S3 Q3

11 Preparation

In geometry the *centroid* of a set of points is, roughly speaking, the average position position of the points. If particles of equal mass were placed at each point, the centroid would coincide with their centre of mass (or centre of gravity). In vector language, if the points have position vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\ldots$, $\mathbf{p}_n$, then the centroid is at $(\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n)/n$.

The three vertices P_1, P_2 and P_3 of an equilateral triangle lie on a circle of radius 1, and their position vectors with respect to an origin at the centre of the circle are $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$.

(i) Explain without calculation, using a symmetry argument, why $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}$.
This takes a bit of thought; or, rather, a bit of thought to come up with a careful explanation. Even if you are not sure about this part, don't let that stop you from continuing to the rest of the question.

(ii) Evaluate $\mathbf{p}_1.\mathbf{p}_1$, $(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3).\mathbf{p}_1$, $\mathbf{p}_2.\mathbf{p}_1 + \mathbf{p}_3.\mathbf{p}_1$ and $\mathbf{p}_2.\mathbf{p}_1$.

(iii) If the vector $\mathbf{x}$ has length k, show that $(\mathbf{x} - \mathbf{p}_1).(\mathbf{x} - \mathbf{p}_1) = k^2 - 2\mathbf{x}.\mathbf{p}_1 + 1$ and find

$$\sum_{i=1}^{3} (\mathbf{x} - \mathbf{p}_i).(\mathbf{x} - \mathbf{p}_i).$$

(iv) Given that P_1 has coordinates $(0, 1)$, use the values of $\mathbf{p}_2.\mathbf{p}_1$ and $\mathbf{p}_2.\mathbf{p}_2$ to find the coordinates of P_2 and P_3.

12 The STEP 3 question

The four vertices P_i $(i = 1, 2, 3, 4)$ of a regular tetrahedron lie on the surface of a sphere with centre at O and of radius 1. The position vector of P_i with respect to O is $\mathbf{p}_i$ $(i = 1, 2, 3, 4)$. Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i \, . \, \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$.

Let X be any point on the surface of the sphere, and let XP_i denote the length of the line joining X and P_i $(i = 1, 2, 3, 4)$.

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}) \, . \, (\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of X with respect to O, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8 \, .$$

(ii) Given that P_1 has coordinates $(0, 0, 1)$ and that the coordinates of P_2 are of the form $(a, 0, b)$, where $a > 0$, show that $a = 2\sqrt{2}/3$ and $b = -1/3$, and find the coordinates of P_3 and P_4.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x} \, . \, \mathbf{p}_i)^2 \, .$$

By letting the coordinates of X be (x, y, z), show further that $\sum\limits_{i=1}^{4} (XP_i)^4$ is independent of the position of X.

Discussion

The very last part is a bit of slog; sorry about that!

It is probably easiest to expand and simplify $\sum\limits_{i=1}^{4} (1 - \mathbf{x} \, . \, \mathbf{p}_i)^2$ before putting in the coordinates of $\mathbf{x}$ and $\mathbf{p}_i$.

You should find that the worst thing you have to calculate is $\left(-\frac{\sqrt{2}}{3} x \pm \frac{\sqrt{2}}{\sqrt{3}} y - \frac{1}{3} z \right)^2$.

2008 S3 Q4

13 Preparation

(i) Sketch the graph $y = \tanh x$.

You can differentiate $\dfrac{\sinh x}{\cosh x}$ to find the gradient. You can also express $\tanh x$ in terms of e^x which will help when explaining the behaviour as $x \to \pm\infty$.

(ii) Show that, for $x \geqslant 0$, $\sqrt{\dfrac{\cosh x - 1}{\cosh x + 1}} = \tanh(x/2)$. What is the corresponding result if $x \leqslant 0$?

(iii) By considering $\cosh y = x$, find $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{arcosh}x$ in terms of x.

(iv) By using the substitution $\sinh y = x$, find $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\mathrm{d}x$ and check your answer by direct integration.

(v) Give a sketch to show that if $\mathrm{f}(t) > \mathrm{g}(t)$ for $t > 0$ and $\mathrm{f}(0) = \mathrm{g}(0)$ then

$$\int_0^x \mathrm{f}(t)\mathrm{d}t > \int_0^x \mathrm{g}(t)\mathrm{d}t$$

and use this result to show that $\frac{1}{2}x^2 > 1 - \cos x$ for $x > 0$.

14 The STEP 3 question

(i) Show, with the aid of a sketch, that $y > \tanh(y/2)$ for $y > 0$ and deduce that

$$\mathrm{arcosh}x > \frac{x-1}{\sqrt{x^2-1}} \quad \text{for} \quad x > 1. \qquad (*)$$

(ii) By integrating $(*)$, show that $\mathrm{arcosh}x > 2\dfrac{x-1}{\sqrt{x^2-1}}$ for $x > 1$.

(iii) Show that $\mathrm{arcosh}x > 3\dfrac{\sqrt{x^2-1}}{x+2}$ for $x > 1$.

[**Note:** $\mathrm{arcosh}x$ is another notation for $\cosh^{-1} x$.]

Discussion

This is a very nice question (where nice is not equivalent to easy), you obtain the first inequality to obtain the second inequality, and the second to obtain the third, and obviously you could keep going, though the denominators get nasty. You are effectively pulling yourself up by your bootstraps.

And just when you feel you can't face another integration, you find that you have already done everything required for part (iii).

But it is not a perfect question, because in fact the inequality of part (ii) could have been obtained from a sketch similar to that of part (i), showing $y/2$ and $\tanh(y/2)$.

2013 S3 Q7

15 Preparation

(i) If $f(0) = 0$ and $f'(x) \geqslant 0$ for $x \geqslant 0$ what can you say about $f(x)$? (Be careful to make an accurate statement!)

(ii) Differentiate with respect to x:

(a) $e^{\sin x}$;

(b) $\cosh(x^2)$

(c) $\sinh(f(x))$, where f is a given function with derivative f$'$;

(d) $xf'(x)$;

(e) $(f'(x))^2$.

(iii) Integrate $Ax \sin x + B \cos x$ with respect to x;

(iv) Integrate $f(x)e^{f(x)}f'(x)$ with respect to x.

(v) By writing the hyperbolic functions in terms of exponentials, show that $\cosh x \geqslant 1$.

16 The STEP 3 question

(i) Let $y(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$ with $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \tfrac{1}{2}y^4 \,.$$

Show by differentiation that E is constant and deduce that $|y(x)| \leqslant 1$ for all x.

(ii) Let $v(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}v}{\mathrm{d}x} + \sinh v = 0$ with $v = \ln 3$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + 2\cosh v \,.$$

Show that $\dfrac{\mathrm{dE}}{\mathrm{d}x} \leqslant 0$ for $x \geqslant 0$ and deduce that $\cosh v(x) \leqslant \frac{5}{3}$ for $x \geqslant 0$.

(iii) Let $w(x)$ be a solution of the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + (5\cosh x - 4\sinh x - 3)\frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) = 0$$

with $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \leqslant \frac{5}{4}$ for $x \geqslant 0$.

Discussion

Don't attempt to solve any of the differential equations in this question! They are all non-linear (i.e. they depend on y^2 or worse), and in general such equations can only be solved numerically by computer.

Even the equation in part (i), which doesn't look too bad, can only be solved in terms of what are called *elliptic* functions.

This is not an easy question, not because any one step is hard, but because there are lots of steps each of which needs an idea which may well be unfamiliar.