STEP Support Programme

STEP 3 Calculus: Hints

1 You are given the derivatives of x and $\arcsin x$ which will be helpful. For the stem you will need to use the chain rule and the quotient rule. The difference of two squares formula might be helpful.

For the two integrals try substituting some values for a and b. Remember that $b > a$.

2 **(i)** If you write $\cosh a$ in terms of e^a and e^{-a} you should then be able to factorise $x^2+2x\cosh a+1$. Remember that e^a is a constant, and that $\displaystyle\int \frac{1}{x+c}\,\mathrm{d}x = \ln|x+c|+k$. A little bit of fiddling at the end is needed to get the required result, and you will need to use the Logarithm Laws quite a bit.

(ii) For both these integrals, start in the same was as part **(i)**, i.e. write $\sinh a$ or $\cosh a$ in terms of exponentials. In both cases some manipulation is needed in order to give the final results in terms of exponentials. A couple of useful results are:

$$\tanh b = \frac{\mathrm{e}^b - \mathrm{e}^{-b}}{\mathrm{e}^b + \mathrm{e}^{-b}}$$

and

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \ (+c)$$

3 In a similar way to integrating $\ln x$ (start by writing $\ln x = 1 \times \ln x$ and then integrate by parts), you should be able to integrate $\dfrac{\ln x}{x}$. Start by writing $I = \displaystyle\int \frac{\ln x}{x}$ and you should end up with I appearing on the right hand side as well (and hopefully not so that it cancels out!).

For the second part there will be more integrating by parts. Be careful with the signs. It will be helpful to note that $\dfrac{\ln x}{x} \to 0$ and $\dfrac{(\ln x)^2}{x} \to 0$ as $x \to \infty$. This can be shown by letting $x = \mathrm{e}^y$ and noting that as $x \to \infty$ we have $y \to \infty$. The expansion $\mathrm{e}^y = 1 + y + \frac{y^2}{2!} + \cdots$ will be useful when doing this.

4 You could start by sketching $y = \dfrac{t}{t+1}$, or by differentiating $\dfrac{t}{t+1}$ to find what the greatest value in the range $0 \leqslant t \leqslant 1$ is. If using the differentiating route for questions like this you should be careful to consider if there are any asymptotes in the range.

For the second relationship between I_{n+1} and I_n try using integration by parts on I_{n+1} bearing in mind you are hoping to get an I_n appearing. You can combine the first two results to get the third result connecting I_{n+1} and I_n.

For the $\ln 2$ relationship, start by simplifying $(I_{n+1} - I_n) + (I_n - I_{n-1}) + \cdots + (I_2 - I_1)$, and calculate what I_1 is.

For the inequalities you can find the lower limit by removing the I_{n+1} term from the previous result. For the upper limit use $I_n > \dfrac{1}{n2^{n-1}}$ in the expression for $\ln 2$.