STEP Support Programme

STEP 3 Hyperbolic Functions: Hints

1 Start by using the substitution $t = \sinh x$. Partial fractions may be useful.

Remember that "Hence" means you should be using your previous work, so try and make this integral look like the previous ones. There is an identity connecting $\sinh^2 x$ and $\cosh^2 x$ which will be useful. Work out the integral to a first and then let $a \to \infty$.

You can write $\cosh x$ and $\sinh x$ in terms of e^x.

2 Some initial thoughts:

- The limits for T and V are the same so substitution may not be the best thing here. Perhaps integration by parts?
- The limits for U and T/V might help find a suitable substitution.
- As ln makes an appearance in several of the integrals, the definitions of the hyperbolic functions in terms of e^x might be useful.

Some useful formulae might be:

$$\cosh^{-1} x = \ln\left[x + \sqrt{x^2-1}\right] \quad \text{where} \quad (x \geqslant 1) \tag{1}$$

$$\sinh^{-1} x = \ln\left[x + \sqrt{x^2+1}\right] \tag{2}$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad \text{where} \quad (|x| < 1) \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}} \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} \tag{6}$$

$$\int \tanh x \,\mathrm{d}x = \ln\cosh x + k \tag{7}$$

When this question was set, these formulae were given in a formula book. In the current exams (2019 onwards) you might be given some useful formulae in the question, such as the expression for $\cosh^{-1} x$, from which you can find the derivative with a little bit of algebraic manipulation.

3 It will help to start by finding y in terms of r, r in terms of θ, $\dfrac{\mathrm{d}r}{\mathrm{d}x}$ in terms of θ and $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$.

Other useful formulae are:

$$\operatorname{cosech}\theta = \frac{1}{\sinh\theta} \tag{8}$$

$$\coth\theta = \frac{\cosh\theta}{\sinh\theta} \tag{9}$$

$$\cosh^2\theta - \sinh^2\theta = 1 \tag{10}$$

$$\coth^2\theta - 1 = \operatorname{cosech}^2\theta \tag{11}$$

$$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B \tag{12}$$

The final proof can be done by induction. See Foundation Assignment 20 for an introduction to proof by induction.

4 Start by substituting $x = 2a\cosh\left(\frac{1}{3}T\right)$ into the given equation.

Next compare $x^3 - 3bx = 2c$ and $x^3 - 3a^2x = 2a^3\cosh T$. Work out some connections between the coefficients, and make sure that these are possible (use the given conditions on b and c!).

You can write T in terms of u and a.

If you know one root of a cubic, you might be able to find the quadratic that the other two roots satisfy.

A bit of knowledge of complex numbers is needed near the end. You need to know that $\mathrm{i}^2 = -1$, $\sqrt{-3} = \mathrm{i}\sqrt{3}$ and if $\omega = \frac{1}{2}\left(-1+\mathrm{i}\sqrt{3}\right)$ then $\omega^2 = \frac{1}{4}\left(1-3-2\mathrm{i}\sqrt{3}\right) = \frac{1}{2}\left(-1-\mathrm{i}\sqrt{3}\right)$.

For the very last part, start by writing down b and c and then work out a, u and $\dfrac{b}{u}$.