Admissions Testing Service

Sixth Term Examination Papers **9475**

MATHEMATICS 3 Afternoon

THURSDAY 23 JUNE 2016 Time: 3 hours

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 8 printed pages and 4 blank pages.

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Section A: Pure Mathematics

1 Let

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2ax + b)^n} \, \mathrm{d}x \,,$$

where a and b are constants with $b > a^2$, and n is a positive integer.

(i) By using the substitution $x + a = \sqrt{b - a^2} \tan u$, or otherwise, show that

$$I_1 = \frac{\pi}{\sqrt{b - a^2}} \,.$$

(ii) Show that $2n(b - a^2)\, I_{n+1} = (2n - 1)\, I_n$.

(iii) Hence prove by induction that

$$I_n = \frac{\pi}{2^{2n-2}(b - a^2)^{n - \frac{1}{2}}} \binom{2n-2}{n-1} .$$

2 The distinct points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ and $R(ar^2, 2ar)$ lie on the parabola $y^2 = 4ax$, where $a > 0$. The points are such that the normal to the parabola at Q and the normal to the parabola at R both pass through P.

(i) Show that $q^2 + qp + 2 = 0$.

(ii) Show that QR passes through a certain point that is independent of the choice of P.

(iii) Let T be the point of intersection of OP and QR, where O is the coordinate origin. Show that T lies on a line that is independent of the choice of P.

Show further that the distance from the x-axis to T is less than $\dfrac{a}{\sqrt{2}}$.

3 **(i)** Given that

$$\int \frac{x^3 - 2}{(x+1)^2}\,\mathrm{e}^x \mathrm{d}x = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\,\mathrm{e}^x + \text{constant}\,,$$

where $\mathrm{P}(x)$and $\mathrm{Q}(x)$ are polynomials, show that $\mathrm{Q}(x)$ has a factor of $x+1$.

Show also that the degree of $\mathrm{P}(x)$ is exactly one more than the degree of $\mathrm{Q}(x)$, and find $\mathrm{P}(x)$ in the case $\mathrm{Q}(x) = x+1$.

(ii) Show that there are no polynomials $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ such that

$$\int \frac{1}{x+1}\,\mathrm{e}^x \mathrm{d}x = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\,\mathrm{e}^x + \text{constant}\,.$$

You need consider only the case when $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ have no common factors.

4 **(i)** By considering $\dfrac{1}{1+x^r} - \dfrac{1}{1+x^{r+1}}$ for $|x| \neq 1$, simplify

$$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})}\,.$$

Show that, for $|x| < 1$,

$$\sum_{r=1}^{\infty} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{x}{1-x^2}\,.$$

(ii) Deduce that

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) = 2\mathrm{e}^{-y}\operatorname{cosech}(2y)$$

for $y > 0$.

Hence simplify

$$\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y)\,,$$

for $y > 0$.

5 **(i)** By considering the binomial expansion of $(1+x)^{2m+1}$, prove that

$$\binom{2m+1}{m} < 2^{2m},$$

for any positive integer m.

(ii) For any positive integers r and s with $r < s$, $P_{r,s}$ is defined as follows: $P_{r,s}$ is the product of all the prime numbers greater than r and less than or equal to s, if there are any such primes numbers; if there are no such primes numbers, then $P_{r,s} = 1$.

For example, $P_{3,7} = 35$, $P_{7,10} = 1$ and $P_{14,18} = 17$.

Show that, for any positive integer m, $P_{m+1\,,\,2m+1}$ divides $\binom{2m+1}{m}$, and deduce that

$$P_{m+1\,,\,2m+1} < 2^{2m}.$$

(iii) Show that, if $P_{1,k} < 4^k$ for $k = 2, 3, \ldots, 2m$, then $P_{1,2m+1} < 4^{2m+1}$.

(iv) Prove that $\mathrm{P}_{1,n} < 4^n$ for $n \geqslant 2$.

6 Show, by finding R and γ, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x+\gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B.

Two curves have equations $y = \operatorname{sech} x$ and $y = a\tanh x + b$, where $a > 0$.

(i) In the case $b > a$, show that if the curves intersect then the x-coordinates of the points of intersection can be written in the form

$$\pm\operatorname{arcosh}\left(\frac{1}{\sqrt{b^2-a^2}}\right) - \operatorname{artanh}\frac{a}{b}.$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y-coordinate of the point of contact in terms of a.

7 Let $\omega = \mathrm{e}^{2\pi \mathrm{i}/n}$, where n is a positive integer. Show that, for any complex number z,

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1\,.$$

The points X_0, X_1, ... , X_{n-1} lie on a circle with centre O and radius 1, and are the vertices of a regular polygon.

(i) The point P is equidistant from X_0 and X_1. Show that, if n is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1\,,$$

where $|PX_k|$ denotes the distance from P to X_k.

Give the corresponding result when n is odd. (There are two cases to consider.)

(ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n\,.$$

8 **(i)** The function f satisfies, for all x, the equation

$$\mathrm{f}(x) + (1-x)\mathrm{f}(-x) = x^2\,.$$

Show that $\mathrm{f}(-x) + (1+x)\mathrm{f}(x) = x^2$. Hence find $\mathrm{f}(x)$ in terms of x. You should verify that your function satisfies the original equation.

(ii) The function K is defined, for $x \neq 1$, by

$$\mathrm{K}(x) = \frac{x+1}{x-1}\,.$$

Show that, for $x \neq 1$, $\mathrm{K}(\mathrm{K}(x)) = x$.

The function g satisfies the equation

$$\mathrm{g}(x) + x\,\mathrm{g}\Big(\frac{x+1}{x-1}\Big) = x \qquad (x \neq 1)\,.$$

Show that, for $x \neq 1$, $\mathrm{g}(x) = \dfrac{2x}{x^2+1}$.

(iii) Find $\mathrm{h}(x)$, for $x \neq 0$, $x \neq 1$, given that

$$\mathrm{h}(x) + \mathrm{h}\Big(\frac{1}{1-x}\Big) = 1 - x - \frac{1}{1-x} \qquad (x \neq 0,\quad x \neq 1)\,.$$

Section B: Mechanics

9 Three pegs P, Q and R are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side $2a$. A particle X of mass m lies on the table. It is attached to the pegs by three springs, PX, QX and RX, each of modulus of elasticity λ and natural length l, where $l < \frac{2}{\sqrt{3}}\,a$. Initially the particle is in equilibrium. Show that the extension in each spring is $\frac{2}{\sqrt{3}}\,a - l$.

The particle is then pulled a small distance directly towards P and released. Show that the tension T in the spring RX is given by

$$T = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2}\ - l\right),$$

where x is the displacement of X from its equilibrium position.

Show further that the particle performs approximate simple harmonic motion with period

$$2\pi\sqrt{\frac{4mla}{3(4a - \sqrt{3}\,l)\lambda}}\ .$$

10 A smooth plane is inclined at an angle α to the horizontal. A particle P of mass m is attached to a fixed point A above the plane by a light inextensible string of length a. The particle rests in equilibrium on the plane, and the string makes an angle β with the plane.

The particle is given a horizontal impulse parallel to the plane so that it has an initial speed of u. Show that the particle will not immediately leave the plane if $ag\cos(\alpha + \beta) > u^2 \tan\beta$.

Show further that a necessary condition for the particle to perform a complete circle whilst in contact with the plane is $6\tan\alpha\tan\beta < 1$.

11 A car of mass m travels along a straight horizontal road with its engine working at a constant rate P. The resistance to its motion is such that the acceleration of the car is zero when it is moving with speed $4U$.

(i) Given that the resistance is proportional to the car's speed, show that the distance X_1 travelled by the car while it accelerates from speed U to speed $2U$, is given by

$$\lambda X_1 = 2\ln\tfrac{9}{5} - 1\,,$$

where $\lambda = P/(16mU^3)$.

(ii) Given instead that the resistance is proportional to the square of the car's speed, show that the distance X_2 travelled by the car while it accelerates from speed U to speed $2U$ is given by

$$\lambda X_2 = \tfrac{4}{3}\ln\tfrac{9}{8}\,.$$

(iii) Given that $3.17 < \ln 24 < 3.18$ and $1.60 < \ln 5 < 1.61$, determine which is the larger of X_1 and X_2.

Section C: Probability and Statistics

12 Let X be a random variable with mean μ and standard deviation σ. *Chebyshev's inequality*, which you may use without proof, is

$$\mathrm{P}\left(|X-\mu| > k\sigma\right) \leqslant \frac{1}{k^2},$$

where k is any positive number.

(i) The probability of a biased coin landing heads up is 0.2. It is thrown $100n$ times, where n is an integer greater than 1. Let α be the probability that the coin lands heads up N times, where $16n \leqslant N \leqslant 24n$.

Use Chebyshev's inequality to show that

$$\alpha \geqslant 1 - \frac{1}{n}.$$

(ii) Use Chebyshev's inequality to show that

$$1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{(2n)!} \geqslant \left(1 - \frac{1}{n}\right)\mathrm{e}^n.$$

13 Given a random variable X with mean μ and standard deviation σ, we define the *kurtosis*, κ, of X by

$$\kappa = \frac{\mathrm{E}\big((X-\mu)^4\big)}{\sigma^4} - 3.$$

Show that the random variable $X - a$, where a is a constant, has the same kurtosis as X.

(i) Show by integration that a random variable which is Normally distributed with mean 0 has kurtosis 0.

(ii) Let $Y_1, Y_2, \ldots, Y_n$ be n independent, identically distributed, random variables with mean 0, and let $T = \sum\limits_{r=1}^{n} Y_r$. Show that

$$\mathrm{E}(T^4) = \sum_{r=1}^{n} \mathrm{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mathrm{E}(Y_s^2)\mathrm{E}(Y_r^2).$$

(iii) Let $X_1, X_2, \ldots, X_n$ be n independent, identically distributed, random variables each with kurtosis κ. Show that the kurtosis of their sum is $\dfrac{\kappa}{n}$.

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